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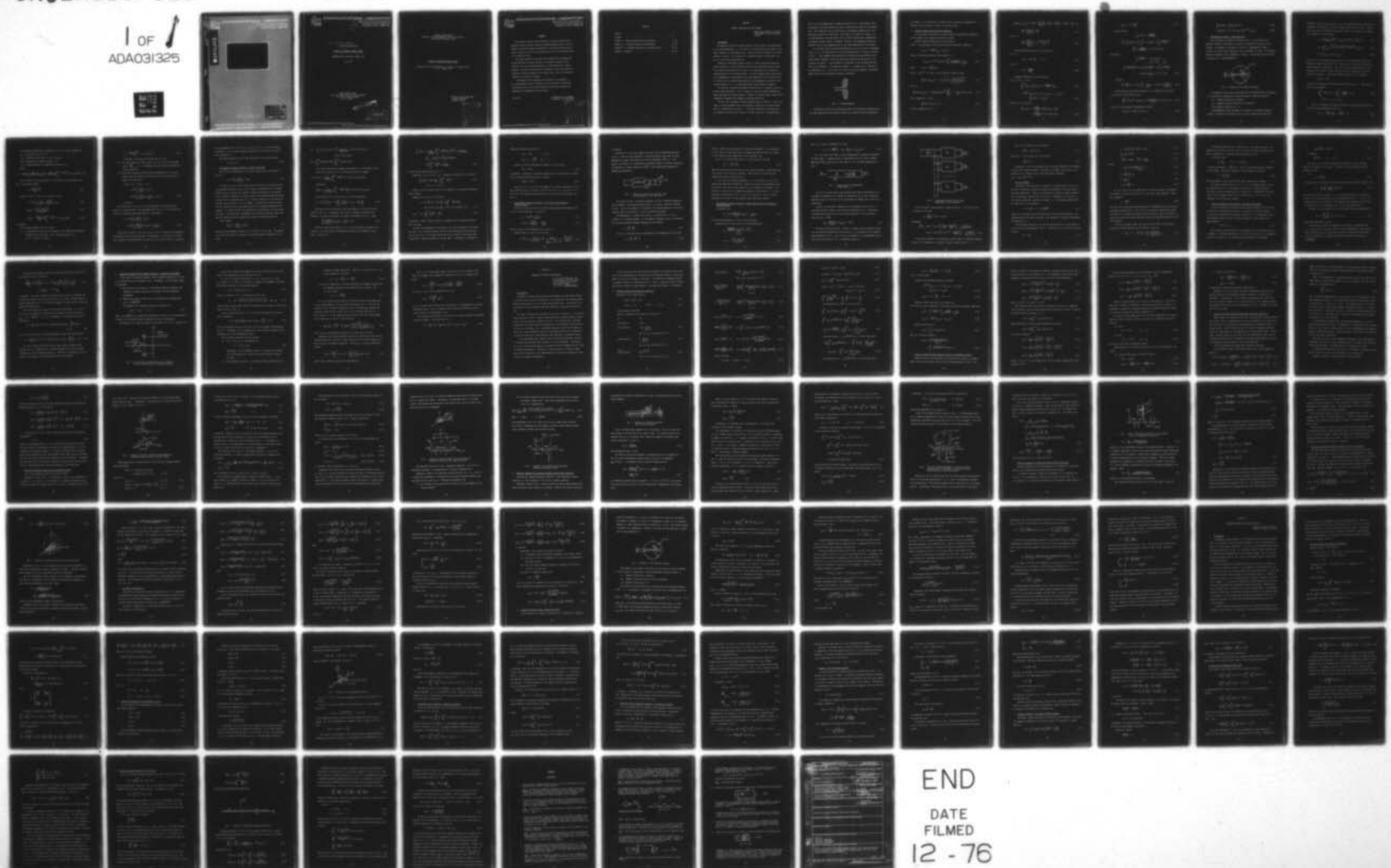

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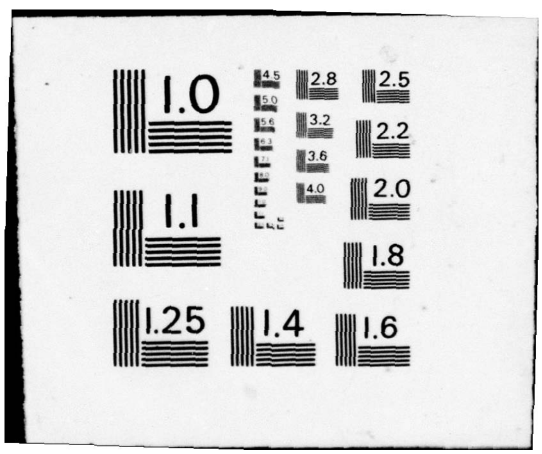
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AFOSR - TR - 76 - 1128

Giorgio Franceschetti

TOPICS IN ADVANCED ANTENNA THEORY

Communications Laboratory Report 76-2

July 1976

Grant AFOSR-72-2263
Air Force Office of Scientific Research
Air Force Systems Command
U.S. Air Force

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Giorgio Franceschetti
Professor of Electromagnetic Fields and Circuits
University of Naples, Italy

TOPICS IN ADVANCED ANTENNA THEORY

Lectures held at the University of Illinois at Chicago Circle
U.S.A., Spring 1976

*"War es ein Gott der diese
Zeichen schriebt?"
(Goethe, Faust)*

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PREFACE

This report contains a major portion of the lecture notes for a graduate course in special topics on advanced antenna theory, that was taught by Professor Giorgio Franceschetti of the University of Naples, Italy, during his visit at the University of Illinois at Chicago Circle in the Spring of 1976.

The three chapters of the report are respectively concerned with antenna admittance and gap problem, with transient radiation from antennas, and with transform properties of radiated fields. Strong emphasis is placed on unconventional techniques. Much of the material represents original research by the Author, and is not yet available in textbooks or scientific journals.

The members of the Communications Laboratory are grateful to Dr. Franceschetti for his lectures as well as for many illuminating discussions, and to the Air Force Office of Scientific Research for supporting the preparation of this report.

July 1976

Piergiorgio L.E. Uslenghi
Director of the Laboratory

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CHAPTER I

ANTENNA ADMITTANCE AND GAP PROBLEM

"Wherein the reader is led down the highways of series manipulation"

1. The problem

An important problem in antenna theory in the RF range is the evaluation of the input admittance, i.e., the ratio of the current to the voltage, at the input terminals of the antenna itself. Motivation of this problem is not only academic in nature - the solution of an elegant boundary value problem - but has also deep roots in applications.

Radiation of broadband signals requires a careful matching between the output admittance of the transmitter and the input admittance of the antenna. For subsurface applications the latter is also depending on the dispersion characteristics of the external medium. In the non-communication applications of electromagnetics, measurement of the input admittance of the antenna, used as a probe, allows possible determination of electromagnetic constants of the external medium, e.g., in plasma applications and biological diagnosis.

The previous discussion highlights the importance of a rigorous theory of antenna input admittance. This is required to give the proper dependence of input admittance on antenna geometry, frequency of applied signal, medium electromagnetic parameters and feeding system characteristics.

The last one is probably the most difficult point to analyze. Let us consider, in the most general case, the antenna as composed by two metal bodies fed by a transmission line (fig. 1). The input admittance at terminals AA' will depend on the position of points (or small areas) AA' on antenna body as

well as on the configuration of feeder structure close to the antenna itself. Accordingly, the input admittance cannot be specified in terms of the antenna only. This difficulty can be alleviated by considering antennas with a symmetry axis and fed at a *small* gap. This leads to the idea of a δ-*gap*, i.e., an infinitesimally thin gap across which a finite voltage (and, consequently, an infinite electric field) is applied. However, this assumption implies an infinite value of the imaginary part of the input admittance of the antenna.

It is obvious that the above difficulties exist also in the zero-frequency limit, i.e., in the definition of the input capacitance of the metal structure of fig. 1. This suggests the possibility of exploring the *gap problem* under quasi-static conditions, where the analytical difficulties connected to the solution are smaller. This procedure is presented in the following Sections, first with reference to the spherical symmetry - a simpler case - and then to the spheroidal one. In the limit of a very thin prolate spheroid, the performance of thin wire cylindrical antenna is recovered.

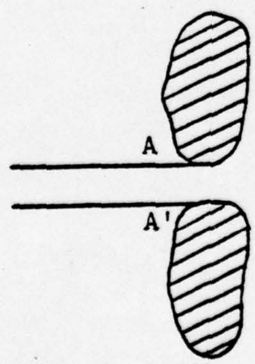


Fig. 1 - Antenna geometry

The analysis which will be presented leads to an analytical expression for the input admittance which clearly exhibits the peculiar resonant performance

of antennas. As an application, transient input responses are computed for antennas in non-dispersive as well as dispersive media.

2. Relevant formulas and pertinent expansions

For the readers' convenience, relevant formulas and pertinent expansions used throughout this chapter are herein collected.

Spherical Hankel functions of the second kind

(note: the superscript (2) has been omitted for notational simplicity)

$$h_{2n-1}(z) = \frac{2n-2}{z} h_{2n-2} - h'_{2n-2}(z) \tag{2.1}$$

where a prime means derivative with respect to z.

$$z\, h_{2n-2}(z) = i^{2n-2} \exp[-iz] \sum_{0}^{2n-2}{}_m \frac{(2n-2+m)!}{m!(2n-2-m)!} \frac{1}{(2iz)^m}, \tag{2.2}$$

where $h_n(z) = \sqrt{\frac{\pi}{2z}}\, H_n(z)$

For $m = (z/2)^{1/3} >> 1$ and $t = [z-(2n-3/2)]/m > 0$ and not large

$$\frac{d}{dz} \ln(z\, h_{2n-2}(z)) \simeq -\frac{1}{m} \exp(i\frac{\pi}{3}) \frac{t\, H_{2/3}(\frac{2}{3} t^{3/2})}{\sqrt{t}\, H_{1/3}(\frac{2}{3} t^{3/2})} \tag{2.3}$$

For $t \to 0$

$$\frac{d}{dz} \ln[z\, h_{2n-2}(z)] \simeq -\frac{1}{m} \exp(i\frac{\pi}{3})(3)^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \simeq -\frac{1}{z^{1/3}}(0.46+i\, 0.80) \tag{2.4}$$

For z large and $z >> 2n-1$

$$\frac{d}{dz} \ln[z\, h_{2n-2}(z)] \simeq -i \tag{2.5}$$

For 2n-1 large and $2n-1 >> z$

$$\frac{d}{dz}\, \ell n[z\, h_{2n-2}(z)] = \frac{2n-1}{z} - \frac{h_{2n-1}(z)}{h_{2n-2}(z)} \simeq \frac{2n-1}{z} - \frac{2(2n-1)}{z} = -\frac{2n-1}{z} \simeq -\frac{2n}{z} \quad (2.6)$$

$$\frac{2n-1}{z}\ \frac{h_{2n-1}(z)}{h_{2n-2}(z)} \simeq \frac{8n^2}{z^2} \quad (2.7)$$

For $z \to 0$

$$\frac{2n-1}{z}\ \frac{h^{(2)}_{2n-1}(z)}{h^{(2)}_{2n-2}(z)} \simeq 2\ \frac{(2n-1)(2n-\frac{3}{2})}{z^2} \quad (2.8)$$

Modified Bessel functions of the first kind

For $t << 1$

$$I_n(t) \simeq \left(\frac{t}{2}\right)^n \frac{1}{n!}, \qquad n \geq 1 \quad (2.9)$$

For $t >> 1$

$$I_n(t) \simeq \frac{\exp(t)}{\sqrt{2\pi t}} \quad (2.10)$$

Legendre polynomials of the first kind

Let $\eta = \cos\theta$, $\alpha = \pi/2-\theta$

$$\int_{-1}^{1} P^1_{2n-1}(\eta)\ P^1_{2m-1}(\eta)\ d\eta = \frac{4n(2n-1)}{4n-1}\ \delta_{nm} \quad (2.11)$$

$$-\sqrt{1-\eta^2}\ \ P^1_{2n-1}(\eta) = 2\ n\ \eta\ P_{2n-1}(\eta) - 2\ n\ P_{2n}(\eta)$$

$$P^1_{2n-1}(0) = 2\ n\ P_{2n}(0) \quad (2.12)$$

For $2n-1 >> 1$, $(2n-1)\theta > 1$

$$P^1_{2n-1}(\eta) \simeq \sqrt{\frac{4n}{\pi\ \sin\theta}}\ \cos\left[(2n-\frac{1}{2})\theta + \frac{\pi}{4}\right] =$$

$$= \sqrt{\frac{4n}{\pi\ \cos\alpha}}\ \cos n\pi\ \cos[(2n-\frac{1}{2})\alpha] \quad (2.13)$$

$$P_{2n}(0) \simeq (-)^n \sqrt{\frac{1}{\pi n}} \tag{2.14}$$

Series summation

$$\sum_{1}^{\infty}{}_{n} \exp(inx) = \frac{\exp(ix)}{1-\exp(ix)}$$

$$\int dx \sum_{1}^{\infty}{}_{n} \exp(inx) = \sum_{1}^{\infty}{}_{n} \int dx \exp(inx) = \frac{1}{i} \sum_{1}^{\infty}{}_{n} \frac{\exp(inx)}{n} =$$

$$= \int dx \frac{\exp(ix)}{1-\exp(ix)} = -\frac{1}{i} \ell n[1-\exp(ix)]$$

Accordingly

$$\sum_{1}^{\infty}{}_{n} \frac{\exp(ix)}{n} = -\ell n[1-\exp(ix)] =$$

$$= -\ell n(2 \sin \frac{x}{2}) - i \frac{x-\pi}{2}$$

$$\sum_{1}^{\infty}{}_{n} \frac{\cos(2n-1/2)x}{n} = \cos \frac{x}{2} \sum_{1}^{\infty}{}_{n} \frac{\cos 2nx}{n} + \sin \frac{x}{2} \sum_{1}^{\infty}{}_{n} \frac{\sin 2nx}{n} =$$

$$= \cos \frac{x}{2} \ell n \frac{1}{2 \sin x} + (\frac{\pi}{2} - x) \sin \frac{x}{2} \tag{2.15}$$

Integrals

$$\int \frac{\sqrt{1-x}}{x} dx = 2\sqrt{1-x} + \int \frac{dx}{x\sqrt{1-x}} = 2\sqrt{1-x} + \ell n \frac{\sqrt{1-x} - 1}{\sqrt{1-x} + 1} \tag{2.16}$$

When the upper limit of the integral is +1, compute the last term of (2.16) for $x = x_0$ and then let $x_0 \to 1$.

$$\int_0^u x^{\nu-1} (u-x)^{\mu-\nu} \exp(xt)\, dx = \frac{\Gamma(\mu-\nu+1)\Gamma(\nu)}{\Gamma(\mu+1)} u^{\mu} \phi(\nu;\mu+1;tu) \tag{2.17}$$

where ϕ is the confluent hypergeometric function

$$\phi(\alpha;\beta;z) = \exp[z]\, \phi(\beta-\alpha;\beta;-z) \tag{2.18}$$

$$\frac{d}{dz}\,\phi(\alpha;\beta;z) = \frac{\alpha}{\beta}\,\phi(\alpha+1;\beta+1;z) \tag{2.19}$$

$$\phi(\alpha;2\alpha;2z) = \Gamma(\alpha+\frac{1}{2})\,\exp(z)\,(\frac{z}{2})^{-\alpha+\frac{1}{2}}\, I_{\alpha-\frac{1}{2}}[z] \tag{2.20}$$

3. The spherical antenna. Input admittance

Let us consider the spherical antenna of fig. 2 to be referred to a spherical system of coordinates $(r,\eta=\cos\theta,\phi)$ centered at the center of the antenna. The antenna is assumed to be fed by a ϕ-independent voltage V_0 $\exp(i\omega t)$ at a small finite equatorial gap of thickness 2s, $s=a\alpha_0$. The external medium is assumed linear, homogeneous, isotropic, time-invariant and described by permittivity ε and permeability μ.

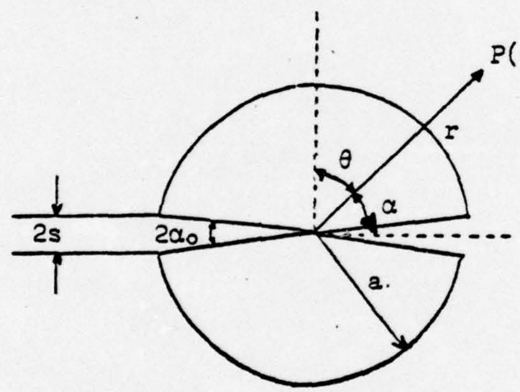


Fig. 2 - Geometry of the spherical antenna

The boundary value problem associated with the radiation from the antenna is the solution of steady-state source-free Maxwell equations subject to

(i) radiation condition at infinity

(ii) symmetry condition relative to ϕ-coordinate

(iii) boundary conditions at $r = a$.

Electromagnetic fields specified in a spherical coordinate system and subject to condition (ii) decouple into two sets of modes characterized by

components (H_r, H_θ, E_ϕ) and (E_r, E_θ, H_ϕ). For the assumed excitation which has no ϕ-component of the electric field, the former modes are not excited. Accordingly, the solution of Maxwell's equations subject to condition (i) in spherical coordinates is the following:

$$E_r = \frac{1}{i\omega\varepsilon\,\sin\theta} \sum_{1}^{\infty}{}_n \frac{2n-1}{r}\, t_n\, h_{2n-1}(kr)\,[P^1_{2n}(\eta) - \eta\, P^1_{2n-1}(\eta)] \tag{3.1}$$

$$E_\theta = \frac{1}{i\omega\varepsilon} \sum_{1}^{\infty}{}_n\, t_n\, \frac{(2n-1)\, h_{2n-1}(kr) - kr\, h_{2n-2}(kr)}{r}\, P^1_{2n-1}(\eta) \tag{3.2}$$

$$H_\phi = \sum_{1}^{\infty}{}_n\, t_n\, h_{2n-1}(kr)\, P^1_{2n-1}(\eta) \tag{3.3}$$

where $h_m(x)$ is the spherical Hankel function of the second kind, $P^1_m(x)$ the associated Legendre polynomial of the first kind, $k = \omega\sqrt{\varepsilon\mu}$, and t_n are as yet unknown expansion coefficients. Note that the usual superscript (2) in the notation of the spherical Hankel function has been omitted for the sake of notational simplicity. Forcing of boundary condition at $r = a$ requires

$$E_\theta(a,\theta) = E(\theta)$$

E being the θ-component of the applied electric field at $r = a$. Note that $E = 0$ save in the gap region,

$$-\int_{-\alpha_0}^{\alpha_0} E(\alpha)\; a\, d\alpha = -\, a\int_{-\eta_0}^{\eta_0} \frac{E(\eta)}{\sqrt{1-\eta^2}}\, d\eta = V_0, \tag{3.4}$$

$$\alpha = \pi/2 - \theta \quad , \quad \eta_0 = \sin\alpha_0 .$$

If $V(\eta)$ is the applied voltage distribution, the applied field distribution $E(\eta)$ follows immediately from (3.4):

$$E(\eta) = -\,\frac{\sqrt{1-\eta^2}}{a}\,\frac{dV}{d\eta} = -\,\frac{\sqrt{1-\eta^2}}{a}\, V'(\eta) \tag{3.5}$$

The expansion coefficients t_n appearing in (3.1-3.3) can be computed by

(i) evaluating (3.2) at $r = a$,

(ii) multiplying both sides of (3.2) by $P^1_{2m-1}(\eta)$

(iii) integrating in η-domain between (-1,1)

(iv) taking advantage of orthogonality relation (2.11).

Hence:

$$t_n = - \frac{i\omega\varepsilon}{(2n-1)\, h_{2n-1}(ka) - ka\, h_{2n-2}(ka)} \, \frac{4n-1}{4n(2n-1)} \int_{-\eta_0}^{\eta_0} \sqrt{1-\eta^2}\, V'(\eta)\, P_{2n-1}(\eta) d\eta \quad (3.6)$$

Let us define the input admittance Y as the ratio of the current at the gap to the applied voltage

$$Y = \frac{2\pi a\, H_\phi(a,\eta_0)}{V_0} \quad (3.7)$$

Using (3.3 and 3.6) into (3.7) the formal result

$$Y = i\pi\omega\varepsilon a \sum_{1}^{\infty}{}_n \frac{4n-1}{(2n-1)^2} \, \frac{R_n(ka)}{R_n(ka)-1} V_n(\eta_0) \quad (3.8)$$

$$R_n(ka) = \frac{(2n-1)\, h_{2n-1}(ka)}{ka\, h_{2n-2}(ka)} \quad (3.9)$$

$$V_n(\eta_0) = - \frac{P^1_{2n-1}(\ _0)}{2\, V_0 n} \int_{-\eta_0}^{\eta_0} \sqrt{1-\eta^2}\, V'(\eta)\, P^1_{2n-1}(\eta) d\eta \quad (3.10)$$

is obtained.

The following comments are now in order.

(i) The expression $V_n(\eta_0)$ can be considered as the normalized excitation voltage of the (2n-1) mode. For a δ-gap excitation, i.e., $V'(\eta) = - V_0 \delta(\eta)$, the result

$$V_n = \frac{[P_{2n-1}(0)]^2}{2n} = 2n\,[P_{2n}(0)]^2 \tag{3.11}$$

is obtained. Note that use has been made of (2.12).

(ii) For large values of n and ka fixed, eqs. (2.7 and 2.14) show that $R_n(ka) \simeq \frac{8n^2}{z^2}$, $V_n \simeq \pi$. Accordingly, the series (3.8) diverges for a δ-gap excitation.

(iii) Since this divergence process does not depend on ka, it is convenient to add and subtract to the series (3.8) its value for ka = 0. Using (2.8), the result

$$Y(ka) = Y(0) + [Y(ka) - Y(0)] =$$

$$= i\omega\pi\varepsilon a \sum_{1}^{\infty}{}_n \frac{4n-1}{(2n-1)^2} V_n(\eta_0) +$$

$$+ i\omega\pi\varepsilon a \sum_{1}^{\infty}{}_n \frac{4n-1}{(2n-1)^2} \frac{1}{R_n(ka)-1} V_n(\eta_0) \tag{3.12}$$

is obtained.

The first series in (3.12) is still diverging for a δ-gap excitation. However, the second series is now converging and, for $\eta_0 << 1$, the δ-gap excitation is a reasonably good approximation. Accordingly,

$$Y = i\omega\pi\varepsilon a \sum_{1}^{\infty}{}_n \frac{4n-1}{(2n-1)^2} V_n(\eta_0) +$$

$$+ i\omega\pi\varepsilon a \sum_{1}^{\infty}{}_n \frac{2n(4n-1)}{(2n-1)^2} [P_{2n}(0)]^2 \frac{1}{R_n(ka)-1} \tag{3.13}$$

Aside from the factor i, the first term in (3.13) is clearly the susceptance associated with the *static capacitance* of the antenna. Its value will, in general, depend on the gap dimension (η_0) and type of excitation ($E(\eta)$). The second

series approaches zero as ω^3 for $\omega \to 0$ (see eq. (2.8)) in a lossless medium. Accordingly, it is a typical dynamical term and will be called the *dynamical input admittance* Y_d.

The formal expression of the input admittance is then the following

$$Y = i\omega\, C_s + Y_d \tag{3.14}$$

4. The spherical antenna continued. Static capacitance

From (3.13) the expression of the static capacitance of the antenna is given by

$$C_s = \pi\varepsilon a \sum_{1}^{\infty}{}_n \frac{4n-1}{(2n-1)^2}\, V_n(\eta_0) \tag{4.1}$$

In order to sum (4.1) the distribution of the field $V'(\eta)$ applied across the gap should be known; this, in turn, requires specification of the feeder geometry and the solution of a generally complicated boundary value problem. It is, therefore, customary to assume *a priori* the voltage distribution across the gap; this is acceptable only if C_s is rather insensitive to the field distribution across the gap. This is, however, a rather crucial point on which the very definition of an input admittance of the considered antenna must be founded. As a matter of fact, this definition is possible only on the assumption that Y is (practically) independent of the field distribution across the gap and therefore does not (practically) need the feeder specification.

Let us define

$$f_n^{\infty}(\eta_0) = \frac{4n-1}{(2n-1)^2}\, V_n^{\infty}(\eta_0), \tag{4.2}$$

where V_n^{∞} is the asymptotic expansion of V_n where n is very large. The expression for f_n^{∞} can easily be computed by using the asymptotic expansion (2.13). Hence:

(i) $P^1_{2n-1}(\eta_0)\ P^1_{2n-1}(\eta_0) \simeq \frac{2n}{\pi}\ \frac{1}{\sqrt{\cos\alpha_0\ \cos\alpha}}\ \{\cos[(2n-\frac{1}{2})(\alpha+\alpha_0)] +$

$$+ \cos[(2n-\tfrac{1}{2})(\alpha-\alpha_0)]\} .$$

(ii) $\int_{-\alpha_0}^{\alpha_0} \cos[m(\alpha-\alpha_0)]d\alpha = \int_{-\alpha_0}^{\alpha_0} \cos[m(\alpha+\alpha_0)]d\alpha;$

this result stems from the change of variable $\alpha \to -\alpha$, and remains valid when an even function of α is introduced inside the integral.

(iii) $V_n(\eta_0) = \frac{-2}{\pi\sqrt{\cos\alpha_0}} \int_{-\alpha_0}^{\alpha_0} \sqrt{\cos\alpha}\ V'(\alpha)\ \cos[(2n-\frac{1}{2})(\alpha+\alpha_0)]d\alpha .$

Accordingly,

$$f_n^\infty(\eta_0) = \frac{-2}{\pi n\sqrt{\cos\alpha_0}} \int_{-\alpha_0}^{\alpha_0} \sqrt{\cos\alpha}\ \frac{V'(\alpha)}{V_0}\ \cos[(2n-\tfrac{1}{2})(\alpha+\alpha_0)]d\alpha .$$

Now the series (4.1) can be recast in the following form:

$$C_s = \pi\varepsilon a \sum_{1}^{\infty}{}_n\ f_n^\infty(\eta_0) + \pi\varepsilon a \left[\sum_{1}^{\infty}{}_n\ \frac{4n-1}{(2n-1)^2}\ V_n(\eta_0) - f_n^\infty(\eta_0)\right] \qquad (4.3)$$

Note that in the second series, each term behaves as $2/\pi n$ plus terms of order $1/n^2$ as $n\to\infty$. Accordingly, the series is uniformly convergent with respect to η_0, and its evaluation can safely be performed for $\eta_0 = 0$. Hence

$$\sum_{1}^{\infty}{}_n \left\{\frac{2n(4n-1)}{(2n-1)^2}\ [P_{2n}(0)]^2 - \frac{2}{\pi n}\right\} \simeq 1.128 \qquad (4.4)$$

The first series which appears in (4.3) is also uniformly convergent with respect to η_0, thus allowing interchange of summation and integration operations. Using (2.15):

$$\sum_{1}^{\infty} f_n^{\infty}(\eta_0) = \frac{-2}{\pi\sqrt{\cos\alpha_0}} \int_{-\alpha_0}^{\alpha_0} \frac{V'(\alpha)}{V_0} \left\{ \cos\frac{\alpha+\alpha_0}{2}\, \ell n\, \frac{1}{2\sin(\alpha+\alpha_0)} \right.$$

$$\left. + \left[\frac{\pi}{2} - (\alpha+\alpha_0)\right] \sin\frac{\alpha+\alpha_0}{2} \right\} \sqrt{\cos\alpha}\, d\alpha \simeq$$

$$\simeq \frac{2}{\pi} \int_{-\alpha_0}^{\alpha_0} \frac{V'(\alpha)}{V_0}\, \ell n\, \frac{1}{2(\alpha+\alpha_0)}\,, \tag{4.5}$$

the last result being valid for $\alpha_0 << 1$.

Assume now $V(-\alpha_0) = 0$; $V(\alpha) = -V_0$. Integration by parts of (4.5) leads to

$$\sum_{1}^{\infty} f_n^{\infty}(\eta_0) = \frac{2}{\pi}\, \ell n\, \frac{1}{4\alpha_0} + \frac{2}{\pi} \int_0^1 \frac{v(x)}{x}\, dx \tag{4.6}$$

$$v(0) = 0 \quad ; \quad v(1) = 1$$

Clearly, only the second term of (4.6) depends on the voltage distribution across the gap.

Summing up (4.4 and 4.5), the result

$$C_s = 2\pi\varepsilon a \left[0.564 + \frac{1}{\pi}\, \ell n\, \frac{a}{4s} + \frac{1}{\pi}\int_0^1 \frac{v(x)}{x}\, dx\right] =$$

$$= 2\pi\varepsilon a\, [0.882 + \frac{1}{\pi}\, \ell n\, \frac{a}{4s} + I(v)] \simeq 2\varepsilon a\, [\ell n\, \frac{4a}{s} + I(v)], \tag{4.7}$$

with $$I(v) = \int_0^1 \left[\frac{v(x)}{x} - 1\right] dx \tag{4.8}$$

is obtained. Clearly, only the term $I(v)$ depends on the voltage distribution across the gap.

An idea of the magnitude of the quantity $I(v)$ can be obtained in two limiting cases. For a constant applied electric field distribution across the gap $v(x) = x$ and $I(v) = 0$. For a gap with sharp rims the electric field should show a square-root singular behaviour at the gap edges. Accordingly, a reasonable

normalized voltage distribution is

$$v(x) = \sqrt{\frac{x}{2}} \qquad , \quad 0 \le x \le \frac{1}{2}$$

$$v(x) = 1 - \sqrt{\frac{1-x}{2}} \quad , \quad \frac{1}{2} \le x \le 1$$

Using (2.16) for evaluating the integral (4.8) the result

$$I(v) = 1.404 \tag{4.9}$$

is obtained. Accordingly, the static capacitance will be practically insensitive to applied field distribution as long as

$$\ell n \frac{4a}{s} >> 1.404 \tag{4.10}$$

For $a/s = 10$, i.e., $\alpha_0 = 5.75°$, the change in the static capacitance in the two considered cases is as high as 32%; for $a/s = 100$, i.e., $\alpha_0 = 0.57°$, it is still 21%. The change becomes negligible for the case of wire antennas (see Sec. 7).

5. The spherical antenna continued. First term of the dynamical input admittance.

From (3.13) the first term of the series representing the dynamical part Y_d of the input admittance Y is given by

$$Y_1 = i\omega\varepsilon a \frac{3\pi}{2} \frac{1}{R_1(ka)-1} \tag{5.1}$$

$$R_1(ka) = \frac{h_1(ka)}{ka\ h_0(ka)} = \frac{1+ika}{(ka)^2} \tag{5.2}$$

where h_1 and h_0 can be computed via (2.1 and 2.2).

Substituting (5.2) into (5.1) the result

$$Y_1 = \frac{3\pi}{2} i\omega\varepsilon a \frac{(ka)^2}{1+ika-(ka)^2} = \frac{3\pi}{2} \frac{(\omega\varepsilon a)^2}{\frac{1}{i\omega\mu a} + \sqrt{\frac{\varepsilon}{\mu}} + i\omega\varepsilon a} = \frac{\frac{3\pi}{2}(\omega\varepsilon a)^2}{\sqrt{\varepsilon/\mu}\ y_1} \tag{5.3}$$

is obtained.

Expression (5.3) has the lumped equivalent circuit representation given in fig. 3, where the box represents a reciprocal gyrator such that the input admittance Y_1 equals Y_0^2 divided by the output admittance $\sqrt{\varepsilon/\mu}\ y_1$. This gyrator is the dual of a conventional transformer wherein the mutual inductance is replaced by a mutual capacitance $\sqrt{3\pi/2}\ \varepsilon a$ and terminating inpedances by terminating admittances.

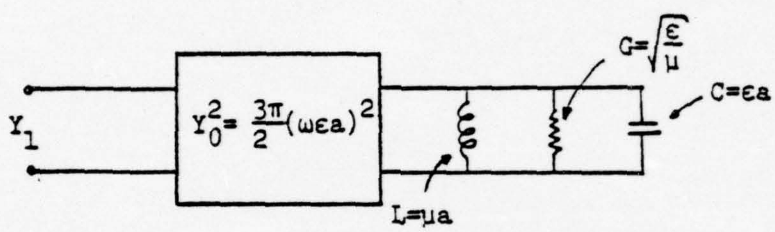


Fig. 3 - Equivalent circuit for the first term of the dynamical input admittance.

The circuit of fig. 3 correctly represents the first resonance properties of the antenna. The normalized admittance y_1 is exactly a parallel resonant circuit. Due to the presence of the gyrator, the network behavior of the first resonant mode of the antenna is a sort of *inverted resonance.*

The admittance Y_1 is purely real when y_1 is purely real, i.e., for $\omega_1\sqrt{\varepsilon\mu}a = 1$ in a lossless medium. At this angular frequency, which is identified as the first *dynamical resonant frequency* of the antenna,

$$Y_1 = \frac{3\pi}{2}\sqrt{\frac{\varepsilon}{\mu}} \quad . \tag{5.4}$$

For $\omega << \omega_1$ the term $1/i\omega\mu a$ is prevaling in the denominator of (5.3) and

$$Y_1 \simeq i\,\frac{3\pi}{2}\sqrt{\frac{\varepsilon}{\mu}}\ z^3 \tag{5.5}$$

where $z = \omega\sqrt{\varepsilon\mu}a$ can be considered as a normalized frequency. The contribution of this mode to the radiation process becomes negligible and the only effect is the lowering of the (capacitive) static reactance $i\omega C_s$.

For $\omega >> \omega_1$ the prevaling terms is, on the contrary, $i\omega\varepsilon a$ and

$$Y_1 \simeq - i \frac{3\pi}{2} \sqrt{\frac{\varepsilon}{\mu}} z \tag{5.6}$$

Again the contribution of this mode to the radiation process is negligible and only reactive local fields are excited. These results are valid also for higher order modes, as shown in Sec. 6.

In the case of a dispersive medium, i.e., $\varepsilon = \varepsilon(\omega)$ and/or $\mu = \mu(\omega)$, the above conclusions must be modified. However, the theory just presented allows for an easy study of the input properties of the antenna in such media. For instance, in the case of an imperfect conductor of conductivity σ, it is only necessary to substitute $\varepsilon + \sigma/i\omega$ to ε into (4.7) and (5.3). This is a great advantage with respect to other antenna input admittance theories.

6. The spherical antenna continued. Higher order terms of the dynamical input admittance

The general term

$$Y_n = i\omega\pi\varepsilon a \frac{2n(4n-1)}{(2n-1)^2} [P_{2n}(0)]^2 \frac{1}{R_n(z)-1} \tag{6.1}$$

of the series appearing in (3.13) is now considered.

Expression (6.1) can be recast in the following form

$$Y_n = \frac{\frac{2n(4n-1)}{(2n-1)^2} \pi [P_{2n}(0)]^2 \omega^2 \varepsilon^2 a^2}{\sqrt{\frac{\varepsilon}{\mu}} y_n} , \tag{6.2}$$

$$y_n = iz + \frac{(2n-1) h_{2n-1}(z)}{i h_{2n-2}(z)} . \tag{6.3}$$

Now (2.1) is used to transform (6.3) into

$$y_n = iz + \frac{(2n-1)^2}{iz} + i(2n-1) \frac{d}{dz} \ell n \; [z \; h_{2n-2}(z)]. \tag{6.4}$$

Eq. (6.4) shows that $\sqrt{\varepsilon/\mu}\; y_n$ can be represented as a parallel connection of three terms: a capacitance εa, an inductance $\mu a/(2n-1)^2$ and a residual admittance $\sqrt{\varepsilon/\mu}\; y_r$ given by the last term of (6.4), as shown graphically in fig. 4.

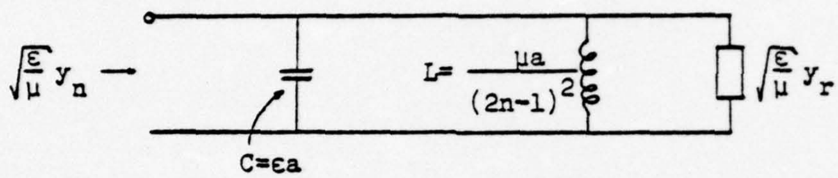


Fig. 4 - Lumped-circuit representation of the term y_n.

Use of (2.2) simply shows that the normalized residual admittance y_r is the parallel connection of a normalized conductance (2n-1) and a normalized admittance given by the ratio of two polynomials of degrees (2n-2) and (2n-1) respectively.

Clearly, the equivalent lumped-circuit representation of (6.2) is a reciprocal gyrator closed on the normalized admittance (6.4). Accordingly, the equivalent circuit for the total input admittance Y of the antenna is given in fig. 5, where

$$Y_{0n}^2 = \frac{2n(4n-1)}{(2n-1)^2} \; \pi[P_{2n}(0)]^2 \; \omega^2 \varepsilon^2 a^2 \tag{6.5}$$

The above results are exact. However, in order to get a physical insight into the resonant properties of the antenna, it is convenient to have approximate expressions for y_n. This, in turn, corresponds to an approximate evaluation of the last term of (6.4) in different ranges of z.

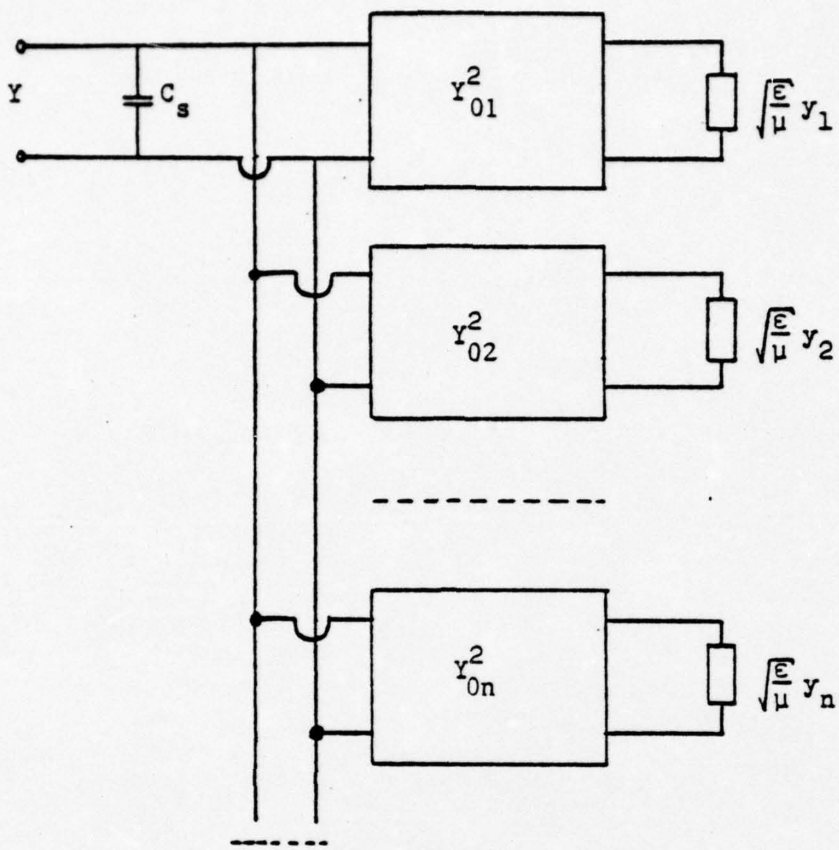


Fig. 5 - Equivalent network for the input admittance Y of the antenna.

We will restrict our attention to large values of z. For $z \simeq 2n-1$, eqs. (2.3 and 2.4) show that

$$y_r \simeq \frac{2n-1}{iz^{1/3}} (0.46 + i\ 0.80).$$

Accordingly

$$\sqrt{\frac{\varepsilon}{\mu}}\, y_n \simeq i\omega\varepsilon a + 0.80 \sqrt{\frac{\varepsilon}{\mu}}\ \frac{2n-1}{z^{1/3}} + \frac{(2n-1)^2}{i\omega\mu a}\left[1 + \frac{0.46}{(2n-1)\, z^{1/3}}\right] \simeq$$

$$\simeq i\omega\varepsilon a + 0.80 \sqrt{\frac{\varepsilon}{\mu}}\ (2n-1)^{2/3} + \frac{(2n-1)^2}{i\omega\mu a}\left[1 + \frac{0.46}{(2n-1)^{4/3}}\right]. \quad (6.6)$$

The network behavior of the antenna is clearly that of a parallel resonant circuit which resonates at a frequency slightly larger than 2n-1.

For $z << 2n-1$, taking (2.6) into account

$$\sqrt{\frac{\varepsilon}{\mu}}\, y_n \quad \frac{2}{iz}\,(2n-1)^2\ , \tag{6.7}$$

while for $z >> 2n-1$, taking (2.5) into account

$$\sqrt{\frac{\varepsilon}{\mu}}\, y_n \simeq iz\ . \tag{6.8}$$

Accordingly, also for the general mode the network behavior of the antenna is that of an inverted resonance. Each mode is strongly excited only in the neighbor of its resonance at the normalized angular frequency $z_{2n-1} = \omega_{2n-1}\sqrt{\varepsilon\mu}\, a \simeq 2n-1$.

7. The wire antenna

The results shown under Secs. 3 through 6 are qualitatively valid also in the case of spheroidal antennas, although the mathematical machinery becomes much more complicated. However, the techniques of *analysis already introduced* can be successfully applied also to this case. Hereafter, only results for *thin* prolate spheroidal antennas are quoted, i.e., for antennas characterized by large values of the parameter

$$\Omega = 2\,\ell n\,\frac{2a}{b} - 2 = 2\,\ell n\,\frac{0.736a}{b} \tag{7.1}$$

where 2a is the larger and 2b the smaller axis of the prolate spheroid. The obtained results can then be applied to wire antennas of length 2a and diameter 2b wherever $\Omega >> 1$.

The equivalent network circuit for the input admittance of the antenna is still of the type represented in fig. 5. Particularly, for the first few terms we have

$$C_s = c_s \varepsilon a, \tag{7.2}$$

$$c_s = 2\pi\left\{\frac{1}{\Omega} + \frac{b}{\pi a}\left[\ell n\,\frac{4a}{s} + I(v)\right]\right\}, \tag{7.3}$$

$$Y_{01}^2 = \frac{3\pi}{2}\ \omega^2\varepsilon^2 a^2 , \tag{7.4}$$

$$\sqrt{\frac{\varepsilon}{\mu}}\,y_1 = i\omega\varepsilon ac + \sqrt{\frac{\varepsilon}{\mu}}\,b + \frac{1}{i\omega\mu a\ell} , \tag{7.5}$$

where:

$$c = \Omega\left[1+0.0826\ \frac{(\Omega+5)}{(\Omega + \frac{5}{11})}\right] \simeq \Omega , \tag{7.6}$$

$$g = \frac{100}{99}\ \frac{25}{11}\ \frac{1}{\psi^2} , \tag{7.7}$$

$$\ell = \frac{11}{25}\ \frac{\psi}{\Omega} , \tag{7.8}$$

$$\psi = \frac{\Omega + 5/11}{\Omega} \simeq 1 . \tag{7.9}$$

Eq. (7.3) shows that the second term in the curly brackets is negligible with respect to the first one when b/a is very small (*thin antennas*). Under this assumption

$$C_s \simeq \frac{2\pi\varepsilon a}{\Omega} \tag{7.10}$$

and this static capacitance becomes practically independent from gap thickness and feeder characteristics. This is the very condition under which an input admittance can be properly defined and also explains the reason why theories which do not take into account the gap problem give acceptable results for thin wire antennas.

In the low frequency range, when only the first mode of the antenna is excited, the input admittance of the antenna is given by

$$Y = i\omega C_s + Y_1 = \sqrt{\frac{\varepsilon}{\mu}}\ \frac{2\pi}{\Omega}\ iz\ \frac{9z + i\Omega\psi[z^2\psi-9]}{9z + i\Omega\psi[(2z)^2\psi-9]} . \tag{7.11}$$

The antenna resonates when Y is purely real. For large values of Ω this happens for $(2z)^2\psi - 9 = 0$ and for $z^2\psi - 9 = 0$, as easily follows by neglecting the term 9z respectively in numerator and denominator of (7.11).

In the first case

$$z_1 = \frac{3}{2\psi} \quad , \quad \text{i.e.,} \quad a \simeq 0.955\,\frac{\lambda}{4} \tag{7.12}$$

λ being the wavelength: $\omega_1\sqrt{\varepsilon\mu} = 2\pi/\lambda$. This is the well-known first resonance of the antenna. The input conductance at this resonance is given by

$$G = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{3\pi}{2}\,\psi \tag{7.13}$$

which correspond to the usual 80 ohm of input resistance at resonance.

The second resonance occurs at $z = 3$, i.e., in a frequency range in which the contribution of the second higher order mode is no longer negligible. Accordingly, previous hypotheses are no longer valid. Furthermore, the conductance at this resonance turns out to be proportional to $1/\Omega^2$, so that this resonance is of no practical interest.

8. Transient response of the antenna. Free-space environment

Let us now consider the transient input current response I(t) of the wire antenna when a time-varying voltage V(t) is applied to the input terminals. It is explicitly assumed that the frequency content of V(t) is negligible in the high frequency range, so that the input admittance Y of the antenna can be approximated by

$$Y \simeq i\omega C_s + Y_1 \, . \tag{8.1}$$

Let ε_0 and μ_0 be the permittivity and permeability of free space, and let $\tau = \sqrt{\varepsilon_0\mu_0}\, a$. If the applied voltage is a pulse of time duration T, its frequency content is of order 1/T. On the other hand, assumption (8.1) requires

$$\omega_{max}\sqrt{\varepsilon_0\mu_0}\, a = \omega_{max}\tau \lesssim 2 . \tag{8.2}$$

Accordingly,

$$\frac{2\pi}{T} \le \frac{2}{\tau} \quad , \quad \text{i.e.,} \quad T \ge \pi\tau . \tag{8.3}$$

Note that 2τ is the time required for the signal to travel a distance equal to the length of the antenna. Accordingly, the pulse duration should exceed this time.

Let

$$p = i\omega\sqrt{\varepsilon_0\mu_0}\, a \tag{8.4}$$

be the analytical continuation of the normalized angular frequency $\omega\sqrt{\varepsilon_0\mu_0}\, a$, and

$$i(t) = \frac{1}{2\pi i}\int_\Gamma \frac{Y(p)}{\sqrt{\varepsilon_0/\mu_0}} \exp(pt)\, dp \tag{8.5}$$

be the normalized unit response (i.e., the current response to a voltage excitation $\delta(t)$) of the approximate circuit corresponding to (8.1). In (8.5), the integration path Γ runs parallel to the imaginary axis leaving to its left all the singularities of the integrand, and t is the time normalized to τ.

The input current response $I(t)$ of the antenna follows by superposition as

$$I(t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{-\infty}^{t} V(t')\, i(t-t')\, dt' , \tag{8.6}$$

where normalized times are used.

In the general case the environment of the antenna is time-dispersive, i.e., $\mu = \mu_0\mu_r(p)$ and $\varepsilon = \varepsilon_0\varepsilon_r(p)$, where the relative permeability $\mu_r(p)$ and the relative permittivity $\varepsilon_r(p)$ describe the dispersive properties of the medium. It is also convenient to introduce the relative refractive index $n_r(p) = \sqrt{\varepsilon_r(p)\mu_r(p)}$.

By substituting the above into (7.11) and neglecting terms of order $1/\Omega^2$ with respect to unity, the result

$$\frac{Y(p)}{\sqrt{\varepsilon_o/\mu_0}} = \frac{\pi}{2\Omega}\,\frac{n_r}{\mu_r}\left[6\alpha_0 + p\,n_r - 3\omega_0^2\,\frac{2\alpha_0 - p\,n_r}{p^2\,n_r^2 + 2\alpha_0\,p\,n_r + \omega_0^2}\right], \tag{8.7}$$

$$\alpha_0 = \frac{9}{8\Omega\,\psi^2}\ , \quad \omega_0 = \frac{3}{2\sqrt{\psi}}$$

is obtained. Note that, insofar as the third term in (8.7) is concerned, the integration contour Γ can be closed by means of a circle of infinite radius in the left-hand plane for $t > 0$, so that the integral is simply given by poles and branch-cut contributions.

Let us now consider the case of a free-space environment, i.e., $n(p) = 1$. In this case no branch-cut contributions are present, but only two poles at $p_{\substack{1\\2}} = -\alpha_0 \pm i\sqrt{\omega_0^2 - \alpha_0^2} \simeq -\alpha_0 \pm i\omega_0$. Accordingly, the integral (8.5) can be evaluated as

$$i(t) = \frac{\pi}{2\Omega}\,\{6\alpha_0\,\delta(t) + \delta'(t) + 3\omega_0^2\,\exp(-\alpha_0 t)\,[\cos\omega_0 t - \frac{3\alpha_0}{\omega_0}\,\sin\omega_0 t],$$

$$(t \geq 0). \tag{8.8}$$

Eq. (8.8) can be recast in the same approximation under the following form

$$i(t) = \frac{\pi}{2\Omega}\left[3\alpha_0\,\delta(t) + \delta'(t) + 3\omega_0^2\,\exp(-\alpha_0 t)\,\cos(\omega_0 t + \frac{3\alpha_0}{\omega_0})\right],\ (t \geq 0). \tag{8.9}$$

Eq. (8.9) clearly shows that the unit input current response of the antenna, for $t > 0$, equals that of a resonant damped circuit with resonance frequency ω_0 and with damping constant α_0. Obviously, the damping of the current has a counterpart in the radiation losses of the antenna.

9. Transient response of the antenna continued. Dispersive environment

In the case of a dispersive medium, the more general expression (8.7) has to be substituted into the integral (8.5). Accordingly, the following changes are generated.

(1) The locations of the poles in the complex p-plane are shifted, this taking into account coupling of the antenna with the dispersive environment.

(2) Branch-cuts should be made, due to the presence of the square-root $n_r(p) = \sqrt{\varepsilon_r(p)\mu_r(p)}$.

For example, for a conducting medium of conductivity σ,

$$\varepsilon_r(p) = 1 + \frac{\gamma}{p} \quad , \quad \mu_r = 1 \tag{9.1}$$

where $\gamma = \sigma a/\sqrt{\varepsilon_0/\mu_0}$ is the normalized conductivity, and now ε_0 is the dielectric constant of the *external medium* (different from that of free-space).

The branch-cut is along the negative α-axis from $\alpha = 0$ to $\alpha = -\gamma$ (see fig. 6).

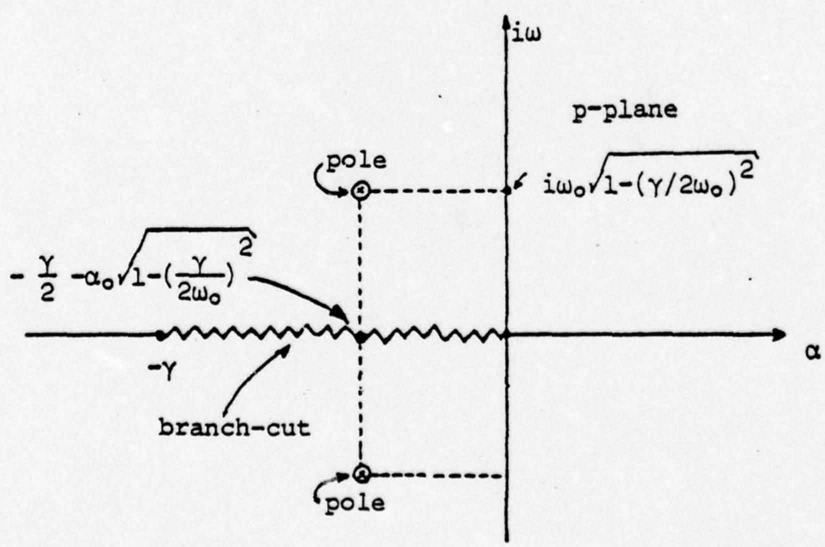


Fig. 6 - Plot of poles and branch-cuts in the complex p-plane for the case of a conductive medium.

Let us first consider what happens to the pole contributions due to the zeros of the denominator of the third term in (8.7).

Let, as in the case of free-space, $p_1 = -\alpha_0 + i\omega_0$, $p_2 = -\alpha_0 - i\omega_0$; $\alpha_0 = 9/8\Omega\ \psi^2$, $\omega_0 = 3/2\sqrt{\psi}$. In the case of a conductive environment, the poles are possible solutions of the following equation

$$(\sqrt{\varepsilon_r}\ p - p_1)(\sqrt{\varepsilon_r}\ p - p_2) = 0, \tag{9.2}$$

where ε_r is given by (9.1). The solutions of (9.2) are

$$p' = -\frac{\gamma}{2} + \sqrt{[-\alpha_0 \pm i(\omega_0 + \frac{\gamma}{2})][-\alpha_0 \pm i(\omega_0 - \frac{\gamma}{2})]} = -\alpha_0' \pm i\omega_0' \tag{9.3}$$

where the plus sign has been chosen for the square-root in order to recover the solution of Sec. 8 for $\gamma = 0$.

The pole contribution $i_p(t)$ to the total current response (8.5) can be easily computed as

$$i_p(t) = \frac{\pi}{2\Omega}\ 3\omega_0^2\ \frac{\omega_0}{\omega_0'}\ \exp(-\alpha_0' t)\ \cos(\omega_0' t + \frac{3\alpha_0}{\omega_0}),\ (t \geq 0), \tag{9.4}$$

which is the generalization to the case $\gamma \neq 0$ of the analogous term appearing in (8.9). Accordingly, ω_0' is recognized as the new resonant frequency of the antenna and α_0' as the new damping constant.

The following observations are now in order.

(i) For $\gamma << 2\omega_0$, eq. (9.2) simplifies to

$$p' = -\frac{\gamma}{2} - \alpha_0 \pm i\omega_0. \tag{9.5}$$

Accordingly, only the damping constant is increased with respect to the free-space case, whereas the resonant frequency remains the same.

(ii) For increasing values of γ the damping further increases and the

resonance frequency diminishes. There is a critical value $\gamma_c = 2\omega_0$ of the conductivity for which

$$p' \simeq -\omega_0 - \sqrt{\alpha_0\omega_0} \pm i\sqrt{\alpha_0\omega_0} . \tag{9.6}$$

(iii) As the conductivity is further increased, the damping increases and the resonant frequency of the antenna is still lowered. For $\gamma >> 2\omega_0$, we have

$$p' = -\gamma \pm i\,\frac{2\alpha_0\omega_0}{\gamma} . \tag{9.7}$$

Let us now consider the branch-cut contributions due to the square-root of $\varepsilon_r(p)$ which appears in the first and third term of (8.7). Note that the closure of the Γ contour of integration in the left-hand plane requires the subtraction from the first term in (8.7) of its limiting value for $|p| \to \infty$. Then, letting $x = -\alpha$, the expression that the sum of these two terms assumes on the upper side of the cut is the following:

$$\frac{\pi}{2\Omega}\,6\alpha_0\left[i\sqrt{\frac{\gamma-x}{x}} - 1\right] - \frac{\pi}{2\Omega}\,3\omega_0^2\,\frac{2\alpha_0 i\sqrt{\frac{\gamma-x}{x}} - (\gamma-x)}{-x(\gamma-x) - 2i\alpha_0\sqrt{x(\gamma-x)} + \omega_0^2} , \tag{9.8}$$

and is equal to the complex conjugate on the lower side of the cut.

Summing up the two contributions along the two sides of the cut and integrating between $(0,-\gamma)$ gives $2\pi i$ times that part of the current response (8.5) which is due to the branch singularity. Letting $i_b(t)$ be such contribution, we have

$$i_b(t) = \frac{3\alpha_0}{\Omega}\int_0^{\gamma}\sqrt{x}\,(\gamma-x)^{3/2}\,\frac{\omega_0^2 + x(\gamma-x)}{[\omega_0^2 - x(\gamma-x)]^2}\exp(-xt)dx \tag{9.9}$$

where terms of order $(\alpha_0/2\omega_0)^2$ have been neglected.

For $\omega_0 > \gamma/2$ we can safely neglect the term $x(\gamma - x)$ with respect to ω_0^2. Then, the integral can be computed by repeated use of (2.17) through (2.20). Hence

$$i_b(t) = \frac{2\pi\alpha_0^2\psi^2}{\omega_0^2} \gamma^3 \exp(-\frac{\gamma t}{2}) \left[\frac{I_1(\frac{\gamma t}{2})}{\gamma t/2} + \frac{I_2(\frac{\gamma t}{2})}{\gamma t/2} \right] \tag{9.10}$$

For $\gamma t/2 >> 1$, eq. (2.10) transforms (9.10) into

$$i_b(t) \sim \frac{4\sqrt{\pi}\ \alpha_0^2\psi^2}{\omega_0^2} (\frac{\gamma}{t})^{3/2} . \tag{9.11}$$

Eq. (9.11) shows that the branch-cut contribution is the dominant term at large observation times. This is a general result whenever the branch-cut has at least one point on the imaginary ω-axis.

The remaining contributions to the total unit current response are straightforward to compute. The final result is:

$$i(t) = \frac{\pi}{2\Omega} \delta'(t) + \frac{\pi}{2\Omega} (6\alpha_0 + \gamma)\delta(t) + i_p(t) + i_b(t). \tag{9.12}$$

CHAPTER II

TRANSIENT RADIATION FROM ANTENNAS

"It is worth emphasizing that our knowledge about the transient response of antennas is very meager indeed"
(T.T. Wu, in Antenna Theory)

1. The problem

Although the steady-state radiation from antennas has been widely studied in the past and the radiative mechanism is well understood, the radiation under transient conditions is still an almost undiscovered field. In spite of this, the problem is rather important from both theoretical and practical points of view.

In order to complete our knowledge in the field of antennas, it is certainly highly desirable to have a complete understanding of radiation properties under steady state *and* transient conditions as well. On the other hand, wide band transmission of signals and pulsed propagation in non-dispersive and dispersive lossless and lossy media (e.g., subsurface radiation, radio links between two satellites in the ionosphere) requires a sharp understanding of the mechanism of transient radiation in non-dispersive and, even more, in *dispersive* media.

It is worth emphasizing, however, that the analytical machinery necessary to correctly describe transient radiation is rather formidable. Accordingly, it is very likely that the physics of radiation could be submerged into an ocean of difficult mathematics. Therefore, it seems attractive to develop simple models which could help in the understanding of the physical mechanism of radiation with a minimum of mathematics.

In the following, first some heuristic procedures are presented which allow for an easy understanding of the radiation from elementary sources, coaxial apertures, wire and loop antennas in free-space. The problem of elementary sources in dispersive media is touched upon as well. Then, a rigorous solution of radiation from an antenna of simple geometry - the spherical antenna - is presented.

2. Relevant formulas and pertinent expansions

Properties of the Dirac function

$$\delta(at+b) = \frac{1}{a}\,\delta(t + \frac{b}{a}) \tag{2.1}$$

$$t\,\dot{\delta}(t) = -\,\delta(t) \quad ; \quad t^2\,\dot{\delta}(t) = -\,2\,t\,\delta(t) \tag{2.2}$$

Inverse Laplace transforms

(Fourier transforms are obtained by letting $p = i\omega$)

$F(p)$	$f(t)$
$F(p)\exp(-p\tau)$	$f(t-\tau)$

$$I_1(px)\exp(-px) \qquad -\frac{1}{\pi x}\,\frac{t}{\sqrt{x^2 - t^2}} \tag{2.3}$$

for $-x \le t \le x$, and zero for $|t| > x$

$$I_0(px)\exp(-px) \qquad \frac{1}{\pi}\,\frac{1}{\sqrt{x^2 - t^2}} \tag{2.4}$$

for $-x \le t \le x$, and zero for $|t| > x$

$$\frac{I_1(px)}{x}\exp(-px) \qquad \frac{1}{\pi x}\sqrt{x^2 - t^2} \tag{2.5}$$

for $-x \le t \le x$, and zero for $|t| > x$

$$I_n(px)\,\exp(-px) \qquad \frac{(-1)^n}{\pi}\,\frac{1}{\sqrt{x^2-t^2}}\cos\left[n\cos^{-1}\frac{t}{x}\right] \tag{2.6}$$

for $-x \le t \le x$, and zero for $|t| > x$

$$\frac{I_n(px)\,\exp(-px)}{(px)^2+a^2} \qquad \frac{(-1)^n}{\pi a x}\int_{-x}^{t}\frac{\cos[n\cos^{-1}u/x]}{\sqrt{x^2-u^2}}\sin\left[\frac{a}{x}(t-u)\right]du \tag{2.7}$$

$-x \le t \le x$

$$px\,\frac{I_n(px)\,\exp(-px)}{(px)^2+a^2} \qquad \frac{(-1)^n}{\pi a x}\int_{-x}^{t}\frac{\cos[n\cos^{-1}u/x]}{\sqrt{x^2-u^2}}\cos\left[\frac{a}{x}(t-u)\right]du \tag{2.8}$$

$-x \le t \le x$

$$\exp[-\tau\sqrt{p^2+x^2}+\tau p] \qquad \delta(t) - \frac{J_1[x\sqrt{t(t+2\tau)}]}{x\sqrt{t(t+2\tau)}}\,x^2\,\tau\,U(t) \tag{2.9}$$

$$\frac{\exp[-\tau\sqrt{p^2+x^2}+\tau p]}{p^2+x^2} \qquad U(t)\int_0^t J_0[x(t-u)]\,J_0[x\sqrt{u(u+2\tau)}\,]\,du \tag{2.10}$$

$$\exp[-\tau\sqrt{p(p+x)}+\tau p] \qquad \delta(t) + \exp\left(\frac{x}{2}t\right)\frac{I_1\left[\frac{x}{2}\sqrt{t(t+2\tau)}\right]}{\frac{x}{2}\sqrt{t(t+2\tau)}}\left(\frac{x}{2}\right)^2\tau \tag{2.11}$$

$$\frac{\exp[-\tau\sqrt{p(p+x)}+\tau p]}{p(p+x)} \qquad U(t)\exp\left[-\frac{xt}{2}\right]\int_0^t I_0\left[\frac{x}{2}(t-u)\right]I_0\left[\frac{x}{2}\sqrt{u(u+2\tau)}\right]du \tag{2.12}$$

Bessel functions

$$J_1(-ipx) = -J_1(ipx) = -i\,I_1(px) \tag{2.13}$$

$$J_0(-ipx) = J_0(ipx) = I_0(px) \tag{2.14}$$

$$J_n(-ipx) = (-)^n J_n(ipx) = \exp(\tfrac{3}{2} in\pi)\, I_n(px) \tag{2.15}$$

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(ix\cos\phi)\, d\phi \tag{2.16}$$

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) = - J_{n+1}(x) + \frac{n}{x} J_n(x) =$$

$$= \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \tag{2.17}$$

$$\int_0^t \frac{J_1(\omega_p\sqrt{2\tau t})}{\omega_p\sqrt{2\tau t}}\, dt = \frac{1}{\omega_p^2\tau}\int_0^x J_1(x)\, dx = \frac{1}{\omega_p^2\tau} \tag{2.18}$$

(The substitution $\omega_p\sqrt{2\tau t} = x$ has been used.)

$$\int_0^t J_0(\omega_p\sqrt{2\tau u})du = \frac{1}{\omega_p^2\tau}\int_0^x x\, J_0(x)dt = \frac{x\, J_1(x)}{\omega_p^2\tau} \quad , \tag{2.19}$$

$$\int_0^t J_0\left(\omega_p\sqrt{u(u+2\tau)}\right)du = \frac{1}{\omega_p}\int_0^x \frac{x J_0(x)}{\sqrt{x^2+\omega_p^2\tau^2}}\, dx =$$

$$= \frac{1}{\omega_p} J_1(x)\,\frac{\sqrt{t(t+2\tau)}}{t+\tau} + \frac{1}{\omega_p}\int_0^x J_1(x)\,\frac{x^2}{(x^2+\omega_p^2\tau^2)^{3/2}}\, dx \tag{2.20}$$

(The substitution $x = \omega_p\sqrt{t(t+2\tau)}$ and (2.17) have been used.)

$$-\tau\frac{\partial}{\partial\tau}\int_0^t J_0\left(\omega_p\sqrt{u(u+2\tau)}\right)du = -\tau\int_0^x J_0'(x)\left[1 - \frac{\omega_p\tau}{\sqrt{x^2+\omega_p^2\tau^2}}\right]dx =$$

$$= -\tau\, J_0(x)\,\frac{t}{t+\tau} + \tau\int_0^x J_0(x)\,\frac{\omega_p\tau\, x}{(x^2+\omega_p^2\tau^2)^{3/2}} \tag{2.21}$$

(The substitution $x = \omega_p\sqrt{t(t+2\tau)}$ and (2.17) have been used.)

$$J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos\left[x - (n + \frac{1}{2}) \frac{\pi}{2}\right] \qquad (2.22)$$

for $x \to \infty$ and n fixed.

Legendre and associated Legendre polynomials

$$-\sqrt{1-\eta^2}\, P^1_{2n-1}(\eta) = 2\, n\, \eta\, P^1_{2n-1}(\eta) - 2\, n\, P_{2n}(\eta)$$

$$P^1_{2n-1}(0) = 2\, n\, P_{2n}(0) \qquad (2.23)$$

$$P_{2n}(0) = (-)^n \sqrt{\frac{1}{\pi n}} \qquad \text{for} \quad n \to \infty \qquad (2.24)$$

Spherical Hankel functions of the second kind

(note that the usual superscript (2) has been omitted for notational simplicity)

$$h_n(x) = i^n \frac{\exp(-ix)}{x} \sum_{0}^{n}{}_m \frac{(n+m)!}{m!(n-m)!} \frac{1}{(2ix)^m} \qquad (2.25)$$

$$h_{2n-1}(z) = \frac{2n-2}{x} h_{2n-2}(z) - h'_{2n-2}(z) \qquad (2.26)$$

Modified Lerch function

$$L(s,q) = \sum_{1}^{\infty}{}_n\, n^{-q} \exp(2\pi ins) \qquad (2.27)$$

Upon use of Mellin transform

$$L(s,q) = \frac{\Gamma(1-q)\exp[i(1-q)\pi/2]}{(2\pi s)^{1-q}} +$$

$$+ \sum_{0}^{\infty}{}_n \frac{(2\pi s)^n\, R(q-n)}{n!} \exp(in\pi/2) \qquad (2.28)$$

3. Transient radiation from elementary electric and magnetic sources

Suppose we have an infinitesimal oscillating electric dipole in a homogeneous isotropic environment, whose Fourier-transformed moment is $\hat{u}_e(\omega)$. The

dipole is located at the origin of a spherical coordinate system (r,θ,ϕ) and is oriented parallel to the direction $\theta = 0$. The components of the dipole's electromagnetic transformed field are given by the well-known expressions

$$\hat{E}_r(\omega) = \zeta \frac{2\,\hat{u}_e(\omega)\,\exp(-i\omega\tau)}{4\pi r} \cos\theta \left[\frac{i\omega}{r} + \frac{c}{r^2} \right] \tag{3.1}$$

$$\hat{E}_\theta(\omega) = \zeta \frac{\hat{u}_e(\omega)\,\exp(-i\omega\tau)}{4\pi r} \sin\theta \left[\frac{(i\omega)^2}{c} + \frac{i\omega}{r} + \frac{c}{r^2} \right] \tag{3.2}$$

$$\hat{H}_\phi(\omega) = \frac{\hat{u}_e(\omega)\,\exp(i\omega\tau)}{4\pi r} \sin\theta \left[\frac{(i\omega)^2}{c} + \frac{i\omega}{r} \right] \tag{3.3}$$

where $\zeta = (\mu/\varepsilon)^{\frac{1}{2}}$, $\tau = r/c$ and $c = 1/(\varepsilon\mu)^{\frac{1}{2}}$.

The transient fields can be easily obtained for the case of a nondispersive environment, when the dipole moment varies arbitrarily in time. By taking the inverse Fourier transform of (3.1)

$$E_r(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{E}_r(\omega)\,\exp(i\omega t)\,d\omega, \tag{3.4}$$

doing the same for the other field components and letting

$$u_e(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_e(\omega)\,\exp(i\omega t)\,d\omega, \tag{3.5}$$

it is easily obtained

$$E_r(t) = \frac{2\zeta}{4\pi r} \cos\theta \left[\frac{\dot{u}_e(t^*)}{r} + \frac{c u_e(t^*)}{r^2} \right] \tag{3.6}$$

$$E_\theta(t) = \frac{\zeta}{4\pi r} \sin\theta \left[\frac{\ddot{u}_e(t^*)}{c} + \frac{\dot{u}_e(t^*)}{r} + \frac{c u_e(t^*)}{r^2} \right] \tag{3.7}$$

$$H_\phi(t) = \frac{1}{4\pi r} \sin\theta \left[\frac{\ddot{u}_e(t^*)}{c} + \frac{\dot{u}_e(t^*)}{r} \right] \tag{3.8}$$

where $t^* = t - r/c = t - \tau$ is the retarded time, and a dot means differentiation with respect to time.

The corresponding expressions for the magnetic dipole are immediately obtained by dulity ($H \to E$, $E \to -H$, $\zeta \to 1/\zeta$, $u_e \to -u_m$). Thus

$$H_r(t) = \frac{2}{4\pi r\zeta}\cos\theta\left[\frac{\dot{u}_m(t^*)}{r} + \frac{cu_m(t^*)}{r^2}\right] \tag{3.9}$$

$$H_\theta(t) = \frac{1}{4\pi r\zeta}\sin\theta\left[\frac{\ddot{u}_m(t^*)}{c} + \frac{\dot{u}_m(t^*)}{r} + \frac{cu_m(t^*)}{r^2}\right] \tag{3.10}$$

$$E_\phi(t) = -\frac{1}{4\pi r}\sin\theta\left[\frac{\ddot{u}_m(t^*)}{c} + \frac{\dot{u}_m(t^*)}{r}\right] \tag{3.11}$$

Assume the applied signal to be a pulse of time duration T. Only the 1/r terms will be shown to be important at distances larger than cT (see Sec. 4). Furthermore, a *physically realizable* dipole can be assumed *elementary* as long as its physical dimensions are small compared to cT.

Let C be the static capacitance and L the static inductance of an electric dipole of effective height h and of a loop of area A, respectively; then the relations

(i) dipole

$$I(t) = C\,\dot{V}(t) \quad , \quad \dot{u}_e = -h\,I \tag{3.12}$$

(ii) loop

$$V(t) = -L\,\dot{I}(t) \quad , \quad u_m = \mu\,A\,I \tag{3.13}$$

are valid in the limit of elementary dipoles.

Substituting (3.12-3.13) into (3.6-3.8) and (3.9-3.11) respectively, the result

(i) Electric short dipole of effective height h:

$$E_\theta(t) = \zeta\,H_\phi(t) \tag{3.14}$$

$$H_\phi(t) = -\frac{\dot{I}(t^*)h}{4\pi rc}\sin\theta = -\frac{Ch\ddot{V}(t^*)}{4\pi rc}\sin\theta \tag{3.15}$$

(ii) Small loop of area A:

$$H_\theta(t) = \frac{\ddot{I}(t^*)A}{4\pi rc^2}\sin\theta = -\frac{\dot{V}(t^*)A}{4\pi rc^2 L}\sin\theta \tag{3.16}$$

$$E_\phi(t) = -\zeta H_\theta(t) \tag{3.17}$$

is obtained for the field at distances larger than cT.

Equations (3.14-3.17) clearly show that the radiation fields are *not* a faithful replica of the transmitted signals (currents or voltages on the radiators). In particular, if the current is proportional to a Dirac pulse $\delta(t)$ for an electric dipole, or to a step function $U(t)$ for a magnetic dipole, the radiation field turns out to be proportional to $\delta(t)$. Therefore, the radiation field *flip-flops*.

4. Radiated power and radiated fields under transient conditions

Under steady-state conditions there is a clear distinction between *radiated and reactive fields*. Radiated fields are the $1/r$ components, and these fields only contribute to the real part of Poynting vector (in frequency-domain) in lossless media. Reactive fields are the $1/r^2$ and $1/r^3$ components and these fields are responsible for the imaginary part of the Poynting vector. Accordingly, radiated fields are associated with the *radiated power* while reactive fields are associated with the imbalance of the mean stored magnetic and electric energies within the period of the oscillating field.

Let us now examine what happens under transient conditions.

For the electric dipole the Poynting vector $\underline{S}$ has r- and θ-components given by

$$S_r(t) = E_\theta(t)H_\phi(t) = \frac{\zeta \sin^2\theta}{(4\pi cr)^2}\left\{\ddot{u}(t) + \frac{c}{r}\frac{\partial}{\partial t}\left[\dot{u}^2 + \frac{c}{r}\dot{u}u + \frac{1}{2}\left(\frac{c}{r}\right)^2 u^2\right]\right\} \tag{4.1}$$

$$S_\theta(t) = -E_r(t)H_\phi(t) = -\frac{\zeta \sin^2\theta}{(4\pi cr)^2}\frac{c}{r}\frac{\partial}{\partial t}\left[\frac{1}{2}\dot{u} + \frac{c}{r}\dot{u}u + \frac{1}{2}\left(\frac{c}{r}\right)^2 u^2\right] \tag{4.2}$$

where the subscript "e" has been dropped. For the magnetic dipole the expressions are the same provided that the magnetic moment is considered and that ζ is replaced by $1/\zeta$.

When a time integration is performed between t_1 and t_2, the total flux of power density within the time interval (t_1, t_2) is obtained. This flux is composed of a term proportional to

$$\int_{t_1}^{t_2} \ddot{u}^2(t)\,dt \tag{4.3}$$

and a second term depending only on the values of u, $\dot{u}$ at the endpoints of the time-integration interval. This second term can always be made equal to zero by choosing $u(t_1) = u(t_2)$; $\dot{u}(t_1) = \dot{u}(t_2)$. For instance, when $u(t)$ is identically zero outside a finite time interval, integration between $(-\infty, +\infty)$ makes this term vanish. Accordingly, this part is identified as the *reversible part of the power*, i.e., that part which is stored in the medium during the radiation process but can be recovered.

On the contrary, the other part of the power proportional to (4.3) is a non-decreasing function of time. Accordingly, it is identified as the *irreversible part of the power*, or *radiated part*, with the understanding that radiation means an irreversible process: the radiated power is absorbed by the sphere at infinity and cannot be recovered.

Note that only the 1/r field components contribute to the radiated power. Accordingly, they will be labeled *radiated fields*. The other fields contribute to the reversible part of the power and will be labeled *local fields*.

A further characterization of the fields is obtained by computing in which region of space each term can be considered as the dominant one.

Reference is made to a Gaussian pulse of electric moment

$$u(t) = u_0 \exp\left[\frac{-2t^2}{T^2}\right] \tag{4.4}$$

T being the effective width of the pulse (distance between inflection points). Substituting (4.4) in (3.6) one obtains

$$E_r(t) = \frac{2\zeta u(t^*)}{4\pi r\ cT^2} \cos\theta \left[-4\ \frac{cT}{r}\ \frac{t^*}{T} + \left(\frac{cT}{r}\right)^2\right] \tag{4.5}$$

$$E_\theta(t) = \frac{\zeta\ u(t^*)}{4\pi r\ cT^2} \sin\theta \left[\left\{\left(4\ \frac{t^*}{T}\right)^2 - 4\right\} - 4\ \frac{cT}{r}\ \frac{t^*}{T} + \left(\frac{cT}{r}\right)^2\right] \tag{4.6}$$

$$H_\phi(t) = \frac{u(t^*)}{4\pi r\ cT^2} \sin\theta \left[\left\{\left(4\ \frac{t^*}{T}\right)^2 - 4\right\} - 4\ \frac{cT}{r}\ \frac{t^*}{T}\right] \tag{4.7}$$

It follows from (4.5-4.7) that the radiation terms are always dominant provided that

$$r >> cT \tag{4.8}$$

except in the neighborhood of $t^* = T/2$, where the radiation terms are zero. Identical results are obtained for the case of a magnetic dipole. It is concluded that the spherical surface $r = cT$ divides the space into two regions, which can be labeled *near space* (inner region) where local fields are dominant and *deep space* (outer region) where radiation fields are dominant. Accordingly, the reversible part of the power is essentially located inside the near space, while the irreversible part resides and propagates in the deep space.

5. Physics of transient radiation from elementary sources

Let us now consider the case of an electric dipole.

Reference is made to Fig. 1. Two charges, +q and -q, are separated by a distance 2ℓ. At a time $t = -\ell/v$ the charge +q is suddenly accelerated and is then made to travel at constant velocity v, so that it collapses on the charge

-q at time $t = \ell/v$. The plot of the electric moment $u(t)$ of the system versus time is given in Fig. 1. Accordingly, a pulsed current is flowing in the time interval $|t| \leq \ell/c$, from $z = \ell$ to $z = -\ell$.

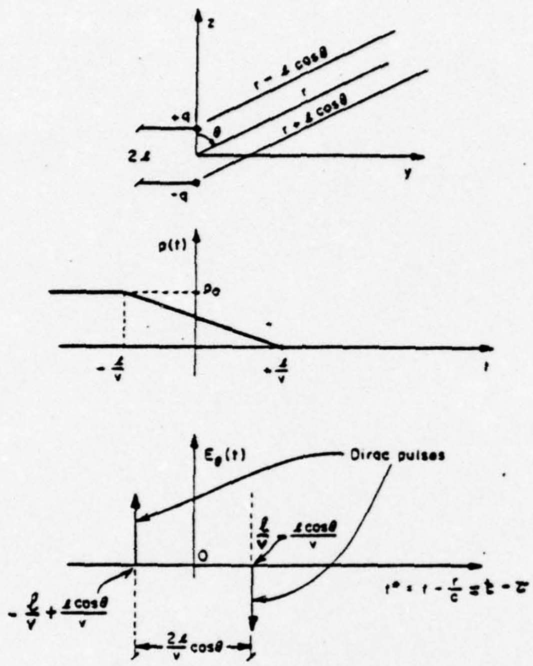


Fig. 1. Relevant to physical model for calculating the transient radiation from an electric dipole.

Formal expressions of current density in the time and frequency domains are the following:

$$J(r,t) = -\hat{z}\, qv\delta(x)\ \delta(y)\ \delta(z+vt)$$

$$= -\hat{\underline{z}}\, q\delta(x)\ \delta(y)\ \delta(t + \frac{z}{v})\ , \qquad |t| \leq \frac{\ell}{v}$$

(see (2.1))

$$J(r,t) = 0\ , \qquad |t| > \frac{\ell}{v} \tag{5.1}$$

$$\hat{J}(r,\omega) = -\hat{\underline{z}}\, q\delta(x)\ \delta(y)\ \exp\left[-i\omega\frac{z}{v}\right], \qquad |z| \leq \ell$$

$$\hat{J}(r,\omega) = 0\ , \qquad |z| > \ell \tag{5.2}$$

$\hat{z}$ being a unit vector in the z direction. The transformed radiation fields associated with (5.2) are

$$\hat{E}_\theta(\omega) = i\zeta \frac{\exp(-i\omega\tau)}{2\pi r} \; q \; \frac{\sin[(\omega\ell/c)(\eta - \cos\theta)]}{\eta - \cos\theta} \sin\theta \tag{5.3}$$

$$\hat{H}_\phi(\omega) = \frac{\hat{E}_\theta(\omega)}{\zeta} \quad , \qquad \eta = \frac{c}{v} \tag{5.4}$$

By inverse Fourier transforming (5.3-5.4) into the time-domain, the result

$$E_\theta(t) = \zeta \frac{q}{4\pi r} \frac{\sin\theta}{\eta+\cos\theta} \cdot [\delta(t^* + x) - \delta(t^* - x)] \tag{5.5}$$

$$H_\phi(t) = \frac{E_\theta}{\zeta} \quad , \qquad x = \frac{\ell}{v} - \frac{\ell}{c}\cos\theta = \frac{\ell}{c}(\eta - \cos\theta) \tag{5.6}$$

is obtained. Equations (5.5-5.6) clearly show that the radiation emanates from the end points of the configuration at times $-\ell/v$ (acceleration) and $+\ell/v$ (deceleration), respectively (see the diagram of Fig. 1). Accordingly, the radiated field is *not* proportional to the time derivative of the current (see (3.14-3.15)) when a time scale less than $2\ell/c$ is considered.

The transition from (5.5-5.6) - *microscopic approach* - to (3.14-3.15) - *macroscopic approach* - is readily accomplished by letting $2\ell/c$ approach zero. One obtains

$$E_\theta(t) = \lim_{2\ell/c \to 0} \zeta \frac{2q\ell}{4\pi rc} \sin\theta \frac{\delta(t^* + x) - \delta(t^* - x)}{2x} = \zeta \frac{u_0}{4\pi rc} \sin\theta \dot{\delta}(t^*) \tag{5.7}$$

$$H_\phi(t) = \zeta E_\theta \tag{5.8}$$

thus recovering (3.14-3.15), since $I(t)2\ell = -q\delta(t)2\ell = -u_0\delta(t)$.

For the case of a magnetic dipole, reference is made to Fig. 2. A constant current I_0 is suddenly switched on at time $t = -t_0$ and switched off at time $t = +t_0$. The current flows in a circular loop of radius R located in the plane (x,y).

Formal expressions for the current in the time and frequency domains are the following:

$$I(t) = I_0[U(t+t_0) - U(t-t_0)] \tag{5.9}$$

$$\hat{I}(\omega) = 2I_0 \frac{\sin\omega t_0}{\omega} \tag{5.10}$$

The transformed radiation fields associated with (5.10) are those of a loop with spatially constant current (5.10). These are known to be

$$\begin{cases} \hat{H}_\theta(\omega) = -\dfrac{RI_0}{rc} \sin(\omega t_0)\; J_1\left(\dfrac{\omega}{c} R \sin\theta\right) \exp(-i\omega\tau) & (5.11) \\[2ex] \hat{E}_\phi(\omega) = -\zeta\, \hat{H}_\theta & (5.12) \end{cases}$$

where $J_1(x)$ is the Bessel function of first kind.

By inverse Fourier transforming (5.11-5.12) into the time-domain the result

$$H_\theta(t) = -\frac{I_0}{2\pi r \sin\theta} [H^{\pm} + H^{\pm}] \tag{5.13}$$

$$E_\phi(t) = -\zeta\, H_\theta \tag{5.14}$$

$$H^{\pm}(t) = \pm \frac{t^* \pm t_0}{[(R\sin\theta/c)^2 - (t^* \pm t_0)^2]^{\frac{1}{2}}}\,, \quad |t \pm t_0| \le \frac{R}{c}\sin\theta \tag{5.15}$$

$$H^{\pm}(t^*) = 0 \qquad |t \pm t_0| > \frac{R}{c}\sin\theta \tag{5.16}$$

is obtained. Use has been made of (2.3 and 2.13).

Equations (5.13-5.16) clearly show that the antenna radiates at times $t = -t_0$, $t = t_0$. For each direction of observation there is continuous radiation for $|t-t_0| \le R\sin\theta/c$, characterized by square-root singularities (see the diagram of Fig. 2). The radiation emanates mainly from points AA' (*flash points*) at the intersections of the loop with the plane containing the direction of

observation and the z axis. No radiation emanates from points BB' (*black points*) due to a cancellation effect. Accordingly, the radiated field is *not* proportional to the second time derivative of the current (see 3.16-3.17) when a time scale less than R/c is considered.

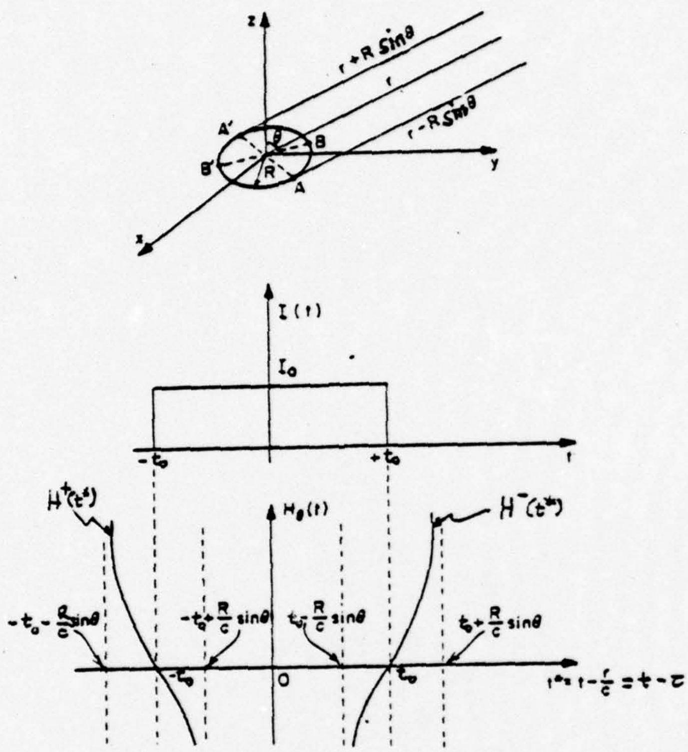


Fig. 2. Relevant to physical model for calculating the transient radiation from a magnetic dipole.

The transition from (5.13-5.16) - *microscopic approach* - to (3.16-3.17) - *macroscopic approach* - is accomplished by letting R/c approach zero.

Reference is made to the radiation which takes place at $t = -t_0$. For $R \to 0$, it seems quite reasonable to substitute for the actual radiated waveform some equivalent pulses (see Fig. 3). Reasonable requirements are:

(i) the strength of each pulse must be equal to the time integral of the actual waveform

(ii) the time allocation of each pulse must divide the actual waveform into parts of equal area. When these requirements are fulfilled, (5.14-5.17) transform into

$$H_\theta(t) = \lim_{R/c\to 0} \frac{I_0R}{2\pi rc} \times \frac{[\delta(t^*+t_0+\frac{3}{4}x)-\delta(t^*+t_0-\frac{3}{4}x)]}{x} = \frac{3}{\pi}\,\frac{I_0\pi R^2}{4\pi rc^2}\sin\theta\,\dot{\delta}(t^*+t_0) \qquad (5.17)$$

$$E_\phi(t) = -\zeta\, H_\theta(t) \quad , \quad x = \frac{R}{c}\sin\theta \qquad (5.18)$$

thus recovering (3.16-3.17), since $I(t) = U(t+t_0)$, apart from the factor $3/\pi = 0.995 \simeq 1$ (depending upon the somewhat arbitrary identification between actual radiated waveforms and equivalent pulses).

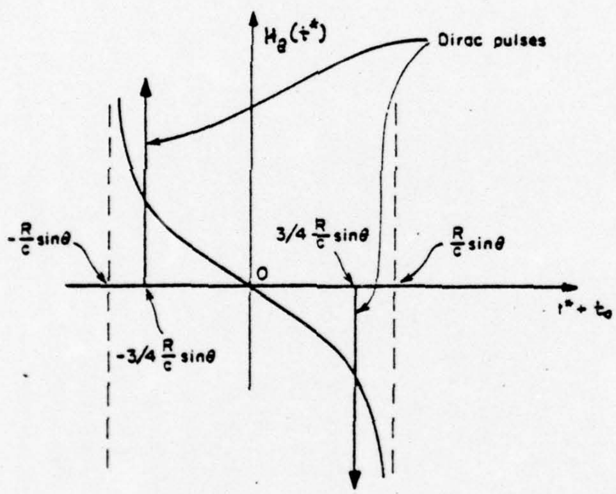


Fig. 3. Relevant to transition from continuous radiation equivalent pulses.

6. Heuristic approach to transient radiation from coaxial apertures

The heuristic approach presented under Sec. 5 for computing transient radiation will now be applied to the case of a coaxial aperture.

Reference is made to Fig. 4 where the radiation from an open ended coaxial cable of circular cross section is considered. However, the results which will

be presented can readily be generalized to apply to radiating apertures of different geometry.

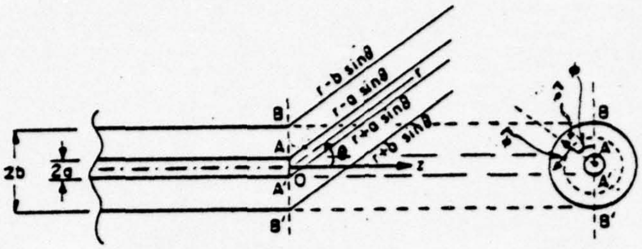


Fig. 4. Relevant to transient radiation from a coaxial aperture.

First, the macroscopic approach will be exploited. Let $V(t)$ be the incident voltage at the open end of the coaxial cable. The electric field in the aperture has only a ρ-component which, under the assumption of perfect open-circuit conditions, is given by

$$E_\rho(t) = \frac{2V(t)}{\rho \ \ln b/a} \tag{6.1}$$

and the magnetic field is zero.

By using the equivalence theorem, the radiated field can be computed as due to an equivalent magnetic current density with only a ϕ-component, $J_{m\phi}(t) = E_\rho(t)$. A z-directed electric dipole of moment given by

$$\begin{aligned} u_e(t) &= \frac{2V(t)\varepsilon}{\ln b/a}\int_a^b \frac{1}{\rho}\pi\rho^2 \, d\rho = \rho\varepsilon \frac{V(t)}{\ln b/a}(b^2 - a^2) \\ &= \frac{A}{2\pi c Z_0} V(t) \end{aligned} \tag{6.2}$$

is therefore associated with the aperture. In (6.2), $A = \pi(b^2 - a^2)$ is the aperture area and $Z_0 = [\zeta \ \ln b/a]/2\pi$ is the characteristic impedance of the coaxial cable.

When A is small compared to $(cT)^2$, the higher-order moments associated with the aperture are negligible, and the radiated field can be obtained from the results of Sec. 3. Hence

$$E_\theta(t) = \frac{\zeta}{Z_0} \frac{A}{2\pi} \frac{\ddot{V}(t^*)}{4\pi r c^2} \sin\theta \tag{6.3}$$

$$H_\phi(t) = \frac{E_\theta(t)}{\zeta} \tag{6.4}$$

Accordingly, the radiated field is proportional to the second time-derivative of the incident voltage.

As a preliminary to the study of the radiation from the microscopic point of view, a model similar to that of Sec. 3 for the electric dipole is considered. A charge q at position $z = -\ell$ is suddenly accelerated at time $t = -z/v$ and is then made to travel at constant velocity v along the positive direction of the z axis. At $t = 0$ and $z = 0$, the charge is suddenly reflected and is again made to travel at constant velocity v along the negative direction of the z axis. At $t = z/v$ and $z = -\ell$ the charge is suddenly stopped.

The radiated field can be computed by following the same procedure as in Sec. 5. Only the results are given in the following. The field is radiated at times $t = -\ell/v$, 0, ℓ/v, from points $z = -\ell$, 0, ℓ, respectively. Particularly, the field radiated at $t = 0$, $z = 0$ (reflection of the charge) is given by

$$E_\theta(t) = -\zeta \frac{q}{2\pi r} \frac{\eta \sin\theta}{\eta^2 - \cos^2\theta} \delta(t^*) \tag{6.5}$$

$$H_\phi(t) = \frac{E_\theta(t)}{\zeta} \quad , \qquad \eta = \frac{c}{v} \tag{6.6}$$

Turning to the coaxial structure of Fig. 4, let $I(t) = q\delta(t-z/c)$ be the current propagating along the cable. The pulsed linear charge densities on the inner and outer conductors will be $q/2\pi a$ and $-q2\pi b$, respectively. These

pulsed charges will apparently propagate with velocity c along the cable. Accordingly, the radiated field can be computed via (6.5) (with $\eta = 1$) and by using superposition. Hence

$$E_\theta(t) = \zeta \frac{q}{(2\pi)^2 r \sin\theta} \cdot \left[\int_0^{2\pi} \delta\left(t - \frac{r_b}{c}\right) d\phi - \int_0^{2\pi} \delta\left(t - \frac{r_a}{c}\right) d\phi \right] \qquad (6.7)$$

where $r_{a,b}$ are the distances of the point (r,θ) from the inner and outer rims of the cable, respectively:

$$r_a \simeq r - a \sin\theta \cos\phi \qquad r_b \simeq r - b \sin\theta \cos\phi \qquad (6.8)$$

In order to calculate the integrals which appear in (6.7), it is convenient to use the Fourier transform

$$\int_{-\infty}^{+\infty} dt \, \exp(-i\omega t) \int_0^{2\pi} \delta(t^* + \frac{b}{c} \sin\theta \cos\phi) \, d\phi$$

$$= \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} \delta(t^* + \frac{b}{c} \sin\theta \cos\phi) \exp(-i\omega t) \, dt$$

$$= \exp(-i\omega\tau) \int_0^{2\pi} d\phi \, \exp(\frac{i\omega}{c} b \sin\theta \cos\phi)$$

$$= 2\pi \exp(-i\omega\tau) J_0(\frac{\omega}{c} b \sin\theta), \qquad (6.8)$$

and similarly for the other integral. Note that use has been made of (2.16). By Fourier inverting (6.8) in the time-domain and substituting in (6.7), the result

$$E_\theta(t) = \zeta \frac{2}{\pi} \frac{q}{4\pi r \sin\theta} [E^b(t^*) - E^a(t^*)] \qquad (6.9)$$

$$H_\phi(t) = \frac{E_\theta(t)}{\zeta} \qquad (6.10)$$

is obtained. In (6.9-6.10) the functions $E^{a,b}(t^*)$ are given by

$$E^x(t) = \frac{1}{[(x \sin\theta/c)^2 - t^2]^{\frac{1}{2}}} \quad , \quad |t| \leq \frac{x}{c} \sin\theta \tag{6.11}$$

$$E^x(t) = 0 \quad , \quad |t| > \frac{x}{c} \sin\theta \tag{6.12}$$

and use has been made of (2.4 and 2.14).

A plot of the radiated field is given in Fig. 5. The preceding results are in complete agreement with those of an exact analysis carried out by using Wiener-Hopf techniques when pulses such that $cT << b$ are considered. This last restriction does not apply in the proximity of the forward direction.

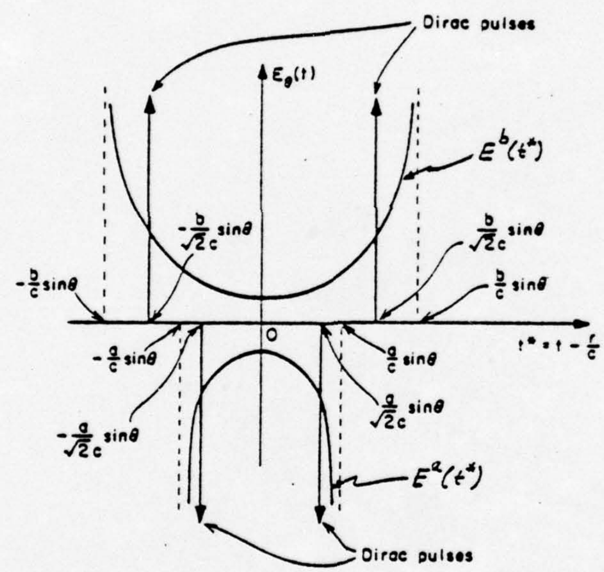


Fig. 5. Plot of the field radiated by a coaxial circular aperture (sum of the negative and positive plots) and transition to equivalent pulses.

Equations (6.8-6.12) clearly show that, for each direction of observation, there is a continuous radiation for $|t^*| \leq b \sin\theta/c$ characterized by square-root singularities. The radiation emanates mainly from points AA', BB' (*flash points*). Accordingly, the radiated field is *not* proportional to the second

time-derivative of the incident voltage $V(t) = Z_0 q\delta(t)\ \bar{V}\delta(t)$ (see (6.3-6.4)), when a time scale less than b/c is considered.

Also in this case the transition from (6.8-6.12) - *microscopic approach* - to (6.3-6.4) - *macroscopic approach* - can be accomplished by substituting for actual radiated waveform equivalent pulses (see Fig. 5) as already done in Sec. 5. Then b/c and a/c are required to approach zero. One obtains (see Fig. 5)

$$E_\theta(t) = \lim_{b/c\to 0;\, a/c\to 0} \frac{\zeta}{Z_0} \frac{\bar{V}}{4\pi r \sin\theta}$$

$$\cdot\left[(x-y)\ \frac{\delta(t^* - x) - \delta(t^* - y)}{x-y} - (x-y)\ \frac{\delta(t^* + y) - \delta(t^* + x)}{x-y}\right]$$

$$= \lim_{b/c\to 0;\, a/c\to 0} \frac{\zeta}{Z_0} \frac{\bar{V}}{4\pi r} \frac{b-a}{\sqrt{2}c}$$

$$\cdot\left[(x+y)\ \frac{\dot{\delta}[t^* - ((x+y)/2)] - \dot{\delta}[t^* + ((x+y)/2)]}{x+y}\right]$$

$$= \frac{\zeta}{Z_0} \frac{A}{2\pi} \frac{\bar{V}\ddot{\delta}(t^*)}{4\pi r c^2} \sin\theta \tag{6.13}$$

$$H_\phi(t) = \frac{E_\theta(t)}{\zeta} \qquad x = \frac{b\sin\theta}{\sqrt{2}c} \qquad y = \frac{a\sin\theta}{\sqrt{2}c} \tag{6.14}$$

thus recovering (6.3-6.4), since $V(t) = \bar{V}\delta(t)$.

7. Heuristic approach to transient radiated from wire antennas

Transient radiation from wire antennas can be calculated by using transmission line techniques for modeling currents and voltages.

A cylindrical antenna of height 2h and diameter 2a is considered (see Fig. 6). If the assumption is made that the antenna is very thin, i.e., $\Omega = 2\ \ell n[2h/a] >> 1$, the current distribution in the frequency domain reduces to

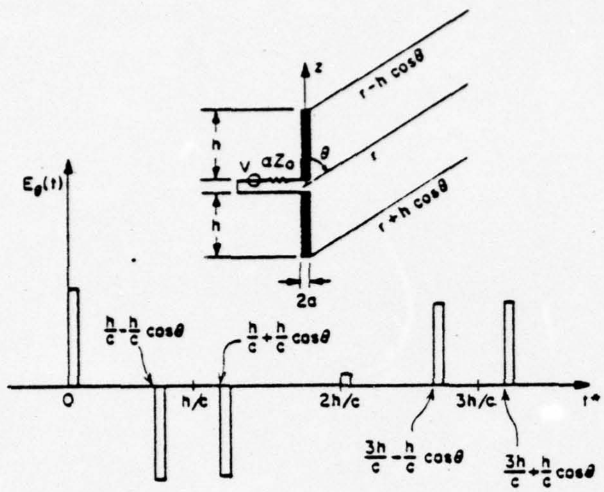


Fig. 6. Plot of the field radiated by a wire antenna fed by a rectangular voltage pulse.

$$\hat{I}(z) = \frac{2\pi}{\zeta\Omega}\hat{V}_{in}\frac{\sin[\omega(h-|z|)/c]}{-i\cos(\omega h/c)} \tag{7.1}$$

where $\hat{V}_{in}$ is the transformed voltage at the antenna terminals. Equation (7.1) is formally equivalent to that of the current distribution along an open-ended transmission line of length h, characteristic impedance $Z_0 = \zeta\Omega/2\pi$ and fed by an ideal voltage generator of zero internal impedance. When the internal impedance of the generator is different from zero and equal to αZ_0 (real), (7.1) can be immediately generalized by using the preceding transmission line analogy. Hence.

$$\hat{I}(z) = \frac{\hat{V}}{Z_0}\;\frac{\sin[\omega(h-|z|)/c]}{\alpha\sin(\omega h/c)-i\cos(\omega h/c)} \tag{7.2}$$

where $\hat{V}$ is the transformed voltage of the generator.

The transformed fields associated with (7.2) are readily seen to be

$$\hat{E}_\theta = \frac{\zeta}{Z_0(\alpha+1)} \hat{V} \frac{\exp(-i\omega r/c)}{\pi r \sin\theta} \cdot \frac{\cos(\omega h/c) - \cos[(\omega/c)h \cos\theta]}{1+\Gamma \exp(-2i\omega h/c)}$$

$$= \frac{\zeta}{Z_0(1+\alpha)} \hat{V} \frac{\exp(-i\omega r/c)}{\pi r \sin\theta} \cdot \{\cos(\omega h/c) - \cos[(\omega/c)h \cos\theta]\} \exp(-i \frac{\omega}{c} h)$$

$$\cdot \sum_{0}^{\infty}{}_n (-\Gamma)^n \exp(-2ni\omega h/c) \tag{7.3}$$

$$\hat{H} = \frac{\hat{E}_\theta}{\zeta} \tag{7.4}$$

where $\Gamma = (1-\alpha)/(1+\alpha)$ is the reflection coefficient at the antenna terminals.

The radiated fields in the time-domain are obtained by inverse Fourier transforming (7.3-7.4). For a Dirac pulse excitation $V(t) = V_0\delta(t)$, $\hat{V} = V_0$ and the radiated fields are given by

$$E_\theta(t) = \frac{\zeta}{Z_0(\alpha+1)} \frac{V_0}{2\pi r \sin\theta}$$

$$\cdot \Big\{ \delta(t^*) + (1-\Gamma) \sum_{0}^{\infty}{}_n (-\Gamma)^n \delta(t^* - \frac{2(n+1)h}{c})$$

$$- \sum_{0}^{\infty}{}_n (-\Gamma)^n [\delta(t^* - \frac{2n+1}{c} (h - h\cos\theta))$$

$$+ \delta(t^* - \frac{2n+1}{c} (h + h\cos\theta))] \Big\} \tag{7.5}$$

$$H_\phi(t) = \frac{E_\theta(t)}{\zeta} \tag{7.6}$$

Equations (7.5-7.6) show that the transient radiation of the antenna consists of a chain of Dirac pulses, each being a replica of the applied signal (save for the amplitude and eventually the sign). The first pulse (first term inside the curly bracket of (7.5)) represents the radiation due to the launching of the current along the antenna at time $t = 0$. The second series in (7.5) represents the radiation from the upper and lower tips of the antenna due to the

charges associated with the current and going back and forth along the antenna.

The first series in (7.5) represents the radiation due to the currents relaunched toward the generator. These currents are absorbed in the internal resistance of the generator itself. When the antenna is short-circuited at its terminals, $\Gamma = 1$ and these last terms disappear.

For an arbitrary applied voltage $V(t)$, the radiated field is easily obtained from (7.5-7.6) by using convolution. A sketch of the transient field radiated by the antenna when the applied voltage is a rectangular pulse is given in Fig. 6.

The above results can easily be applied to the case of a monopole antenna on an infinite perfectly conducting ground plane. The radiation from the lower tip of the antenna disappears, but the transient pulses reflected from the ground plane must be taken into account. In this way, a very satisfactory agreement with experimental results is obtained.

8. Heuristic approach for transient radiation from loop antennas

The transient radiation from a circular loop can be obtained following similar lines. The loop of radius R is fed by a voltage generator of internal impedance αZ_0 (see Fig. 7). The transformed radiation field due to a traveling-wave current $I_0 \exp[-is\chi]$ along the loop is known to be [see S.M. Prasad and B.N. Das, "A circular loop antenna with travelling-wave current distribution," IEEE Trans. Antennas Propagat., Vol. AP-18, No. 2, pp. 228-280, 1970]

$$\hat{E}_\phi = E_1 \left\{ - \frac{J_1(\omega R \sin\theta/c)}{2s} + \sum_{1}^{\infty}{}_n \frac{(i)^n J_n'(\omega R \sin\theta/c)}{s^2 - n^2} (s \cos n\phi - in \sin n\phi) \right\} \tag{8.1}$$

$$\hat{E}_\theta = E_1 \cos\theta \sum_{1}^{\infty}{}_n \frac{(i)^n \, n \, J_n(\omega R \sin\theta/c)}{(s^2 - n^2)(\omega R \sin\theta/c)} \cdot (s \sin n\phi + in \cos n\phi) \tag{8.2}$$

$$\hat{H} = \frac{\hat{\underline{r}} \times \underline{E}}{\zeta} \tag{8.3}$$

where

$$E_1 = - i\zeta \frac{I_0 R\omega}{2\pi r c} [\exp(-i2\pi s) - 1] \exp[-i\omega\tau] \tag{8.4}$$

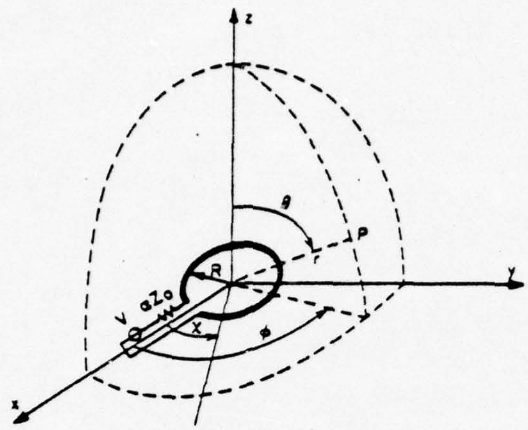


Fig. 7. Relevant to transient radiation from circular loop.

Expressions (8.1-8.4) are the generalization of the field radiated by a loop with a traveling-wave current $I_0 \exp[im\chi]$, m integer. Letting $s = m+\xi$ and then ξ approach zero, the conventional results are recovered.

Now, the transformed current along the loop is assumed to be coincident with that of a transmission line of length πR, characteristic impedance Z_0 (real) and short-circuited at its end. Hence

$$\hat{I} = I_0 \frac{\cos[\omega R(\pi - |\chi|)/c]}{\cos(\omega\pi R/c)}$$

$$= \frac{\hat{V}}{Z_0} \frac{\cos[\omega R(\pi - |\chi|)/c]}{\alpha\cos(\omega\pi R/c) + i \sin(\omega\pi R/c)} \tag{8.5}$$

where $\hat{V}$ is the transformed voltage of the generator.

Expression (8.5) can be recast in terms of two traveling-wave currents moving in the positive and negative χ angular coordinate directions. By algebraic manipulation

$$\hat{I} = \frac{\hat{V}}{Z_0(\alpha+1)} \; \frac{\exp(-i\omega R\chi/c) + \exp[-i\omega R(2\pi - \chi)/c]}{1 - \Gamma \exp(-2\pi i\omega R/c)} \tag{8.6}$$

From (8.6) and (8.1-8.4) with $s = \omega R/c$, and using superposition, the transformed field radiated by the loop can be computed. In doing that, note that for the second traveling wave $s = \omega R/c$ changes into $-\omega R/c$ and E_1 into $-E_1$. Hence

$$\hat{E}_\phi = E_0 \left\{ - \frac{J_1(\omega R \sin\theta/c)}{\omega R/c} + 2 \frac{\omega}{c} R \cdot \sum_{1}^{\infty}{}_n \frac{(i)^n J_n'(\omega R \sin\theta/c)}{(\omega R/c)^2 - n^2} \cos n\phi \right\} \tag{8.7}$$

$$\hat{E}_\theta = E_0 \frac{2 \cos\theta}{\sin\theta} \sum_{1}{}_n \frac{(i)^n nJ_n(\omega R \sin\theta/c)}{(\omega R/c)^2 - n^2} \sin n\phi \tag{8.8}$$

$$\hat{H} = \frac{\hat{\underline{r}} \times \hat{\underline{E}}}{\zeta} \tag{8.9}$$

where

$$E_0 = - i \frac{\zeta V}{Z_0(\alpha+1)} \frac{R}{2\pi rc} [\exp(-2\pi i\omega R/c) - 1] \exp[-i\omega\tau] \sum_{1}^{\infty}{}_n \Gamma^n \exp(-2n\pi i\omega R/c) \tag{8.10}$$

The solution for the transient radiated field is obtained by Fourier transforming (8.10). In the case of a Dirac pulse excitation $V(t) = V_0\delta(t)$, and the transform corresponds to the field radiated by a point charge going around in the loop. This transform can be easily computed by using results (2.5, 2.7, 2.8, 2.15, 2.17).

9. The effect of dispersion

Assume now that the electric dipole considered in Sec. 3 is surrounded by a homogeneous isotropic dispersive environment characterized by relative permittivity $\varepsilon_r(\omega)$ and relative permeability $\mu_r(\omega)$. It is also convenient to introduce the relative refraction index $n_r(\omega) = \sqrt{\varepsilon_r(\omega)\mu_r(\omega)}$.

Letting $\zeta_0 = \sqrt{\mu_0/\varepsilon_0}$ and $p = i\omega$, the generalization of eqs. (3.1-3.3) to this situation is the following

$$\hat{E}_r(p) = \zeta_0 \frac{2\,\mu_r u(p)\exp(-pn_r\tau)\,\cos\theta}{4\pi r}\left[\frac{p}{rn_r} + \frac{c}{r^2 n_r^2}\right], \tag{9.1}$$

$$\hat{E}_\theta(p) = \zeta_0 \frac{\mu_r u(p)\exp(-pn_r\tau)\,\sin\theta}{4\pi r}\left[\frac{p^2}{c} + \frac{p}{rn_r} + \frac{c}{r^2 n_r^2}\right], \tag{9.2}$$

$$\hat{H}_\phi(p) = \frac{u(p)\exp(-pn_r\tau)\,\sin\theta}{4\pi r}\left[\frac{p^2 n_r}{c} + \frac{p}{r}\right]. \tag{9.3}$$

Assume now $\mu_r = 1$. Relations (9.1-9.3) can be recast under the following form

$$\hat{E}_r(p) = \zeta_0 \frac{2\exp(-p\tau)\,\cos\theta}{4\pi r}\left[\frac{p}{r}\,F_2(p) + \frac{c}{r^2}\left(1 - \tau\frac{\partial}{\partial\tau}\right)F_2(p)\right], \tag{9.4}$$

$$\hat{E}_\theta(p) = \zeta_0 \frac{\exp(-p\tau)\sin\theta}{4\pi r}\left[\frac{p^2}{c}\,F_1(p) + \frac{p}{r}\,F_2(p) + \frac{c}{r^2}\left(1 - \tau\frac{\partial}{\partial\tau}\right)F_2(p)\right], \tag{9.5}$$

$$\hat{H}_\phi(p) = \frac{\exp(-p\tau)\sin\theta}{4\pi r}\left[\frac{p^2}{c}\,F_1(p) + \frac{p}{r}\left(1 - \tau\frac{\partial}{\partial\tau}\right)F_1(p)\right], \tag{9.6}$$

where

$$F_1(p) = u(p)\,\exp(-p\tau[n_r(p) - 1]), \tag{9.7}$$

$$F_2(p) = u(p)\,\frac{\exp(-p\tau[n_r(p) - 1])}{n_r^2(p)}. \tag{9.8}$$

Eqs. (9.4-9.6) show that the transient radiation from the dipole can be expressed in terms of the transforms of the two functions (9.7-9.8).

Let us now assume $u(p) = u_0$, and consider the case of a collisionless cold plasma for which

$$n_p(p) = \sqrt{1 + \frac{\omega_p^2}{p^2}}. \tag{9.9}$$

By using results (2.9-2.10), the transient radiated field can be cast under the following form

$$E_r(t) = \zeta_0 \frac{2\,u_0 \cos\theta}{4\pi rc}\left[\frac{1}{\tau}\,\frac{\partial^3 f_2}{\partial t^{*3}} + \frac{1}{\tau^2}\,\frac{\partial^2}{\partial t^{*2}}\left(1 - \tau\frac{\partial}{\partial \tau}\right)f_2\right], \tag{9.10}$$

$$E_\theta(t) = \zeta_0 \frac{u_0 \sin\theta}{4\pi rc}\left[\frac{\partial^2 f_1}{\partial t^{*2}} + \frac{1}{\tau}\,\frac{\partial^3 f_2}{\partial t^{*3}} + \frac{1}{\tau^2}\,\frac{\partial^2}{\partial t^{*2}}\left(1 - \tau\frac{\partial}{\partial \tau}\right)f_2\right], \tag{9.11}$$

$$H_\phi(t) = \frac{u_0 \sin\theta}{4\pi rc}\left[\frac{\partial^2 f_1}{\partial t^{*2}} + \frac{1}{\tau}\,\frac{\partial}{\partial t^*}\left(1 - \tau\frac{\partial}{\partial \tau}\right)f_1\right], \tag{9.12}$$

where

$$f_1(t) = \delta(t) - \omega_p^2\tau\,\frac{J_1\left(\omega_p\sqrt{t(t+2\tau)}\right)}{\omega_p\sqrt{t(t+2\tau)}}\,U(t) \tag{9.13}$$

$$f_2(t) = U(t)\int_0^t J_0[\omega_p(t-u)]\,J_0[\omega_p\sqrt{u(u+2\tau)}]\,du \tag{9.14}$$

It is noted that the signal is identically zero for $t < t^* = t - \tau = t - r/c$, which is the usual causality condition.

The *early time* behavior of the field is now studied at large distances from the source. For $t \to 0$, $\omega_p\tau \to \infty$, the function

$$\frac{J_1\left(\omega_p\sqrt{t(t+2\tau)}\right)}{\omega_p\sqrt{t(t+2\tau)}} \simeq \frac{J_1\left(\omega_p\sqrt{2\tau t}\right)}{\omega_p\sqrt{2\tau t}} \tag{9.15}$$

tends to coalesce toward $t = 0$. Use of (2.18) shows that the time integral from $-\infty$ up to $+\infty$ equals $1/\omega_p^2\tau$. Accordingly, it is reasonable to substitute to (9.15) a Dirac pulse of amplitude $1/\omega_p^2\tau$ and centered at the *center of gravity* of the area of (9.15) (as already done in Sec. 5). This center of gravity is readily computed from (2.18) to be equal to 3/2. Accordingly

$$f_1(t) \simeq \delta(t) - \delta\left(t - \frac{9}{8\omega_p^2\tau}\right) \simeq \frac{9}{8\omega_p^2\tau}\,\dot{\delta}(t) \tag{9.16}$$

for $\omega_p\tau \to \infty$.

Let us now consider the function $f_2(t)$. For $t \to 0$, $\omega_p \tau \to \infty$

$$f_2(t) \simeq \int_0^t J_0\left(\omega_p \sqrt{2\tau u}\right) du = 2t \frac{J_1\left(\omega_p \sqrt{2\tau t}\right)}{\omega_p \sqrt{2\tau t}} \tag{9.17}$$

where use has been made of (2.19). Again the function $J_1(x)/x$ approaches a Dirac pulse as $\omega_p \tau \to \infty$. Accordingly

$$f_2(t) \simeq \frac{2}{\omega_p^2 \tau}\, t\, \delta\left(t - \frac{9}{8\omega_p^2 \tau}\right) \tag{9.18}$$

Substituting (9.16-9.17) in (9.10-9.12) and taking into account (2.2), the result

$$\begin{cases} E_\theta(t) = \zeta_0 \dfrac{\mu_0 \sin\theta}{4\pi r c} \dfrac{9}{8\omega_p^2 \tau} \dddot{\delta}(t^*) & (9.19) \\[2ex] H_\phi(t) = \dfrac{E_\theta(t)}{\zeta_0} & (9.20) \end{cases}$$

is obtained for $t \to 0$, $\omega_p \tau \to \infty$. Accordingly, only the *radiative components* of the field contribute to its early time behavior, and the field tend to be vanishingly small for $\omega_p \tau \to \infty$.

The *late time behavior* of the field is now studied at large distances from the sources, i.e., in the limit $\omega_p t^* \to \infty$, $\omega_p \tau \to \infty$, $t >> \tau$. From (2.20-2.21) it follows that

$$f_2(t) \simeq \frac{1}{\omega_p} J_0(\omega_p t)\, J_1(\omega_p t) \tag{9.21}$$

$$-\tau \frac{\partial}{\partial \tau} f_2(t) \simeq -\tau J_0^2(\omega_p t) \tag{9.22}$$

Substituting in (9.10-9.12) and using (2.22) the result

$$E_r(t) \simeq \zeta_0 \frac{2\mu_0 \cos\theta\, \omega_p^3}{4\pi rc} \sqrt{\frac{2}{\pi\omega_p t}} \left[-\sqrt{\frac{2}{\pi}}\, \frac{2}{\omega_p \tau}\, \frac{\sin 2\omega_p t}{\sqrt{\omega_p t}} \right] \qquad (9.23)$$

$$E_\theta(t) \simeq \zeta_0 \frac{\mu_0 \sin\theta\, \omega_p^3}{4\pi rc} \sqrt{\frac{2}{\pi\omega_p t}} \left[\frac{\tau}{t} \cos\left(\omega_p t + \frac{\pi}{4}\right) - \sqrt{\frac{2}{\pi}}\, \frac{2}{\omega_p \tau}\, \frac{\sin 2\omega_p t}{\sqrt{\omega_p t}} \right] \qquad (9.24)$$

$$H_\phi(t) \simeq \frac{\tau}{t} \frac{\mu_0 \sin\theta\, \omega_p^3}{4\pi rc} \sqrt{\frac{2}{\pi\omega_p t}} \left[\frac{\tau}{t} \cos\left(\omega_p t + \frac{\pi}{4}\right) + \frac{\cos(\omega_p t - \frac{\pi}{4})}{\omega_p \tau} \right] \qquad (9.25)$$

is obtained.

Accordingly, the following conclusions are drawn:

(i) The magnetic field is negligible compared to the electric field.

(ii) Both the $1/r$ and $1/r^2$ components play a role in the expression of the electric field.

(iii) One or the other component dominates according to the relative values of τ/t and $1/\omega_p\tau$.

The case of a conductive medium with

$$n_r(p) = \sqrt{\frac{p+\gamma}{p}} \qquad (9.26)$$

with $\gamma = \sigma/\epsilon_0$ and σ the conductivity, can be analyzed in a similar way. Transient radiated field expressions are similar to (9.12) with

$$f_1(t) = \delta(t) + \left(\frac{\gamma}{2}\right)^2 \tau \frac{I_1(\frac{\gamma}{2}\sqrt{t(t+2\tau)})}{\frac{\gamma}{2}\sqrt{t(t+2\tau)}} \exp\left(\frac{\gamma}{2} t\right), \qquad (9.27)$$

$$f_2(t) = \exp\left[-\frac{\gamma}{2} t\right] \int_0^t I_0\left[\frac{\gamma}{2}(t-u)\right] I_0\left[\frac{\gamma}{2}\sqrt{u(u+2\tau)}\right]. \qquad (9.28)$$

10. Transient radiation from a spherical antenna

Let us consider the spherical antenna of Fig. 8 referred to a spherical

system of coordinates $(r, \eta = \cos\theta, \phi)$ centered at the center of the antenna. The antenna is assumed to be fed by a ϕ-independent voltage $V(\omega)$ (in frequency-domain) at a small equatorial gap of thickness 2s, $s = a\,\alpha_0$. The external medium is assumed to be homogeneous, isotropic, non-dispersive and described by permittivity ε and permeability μ.

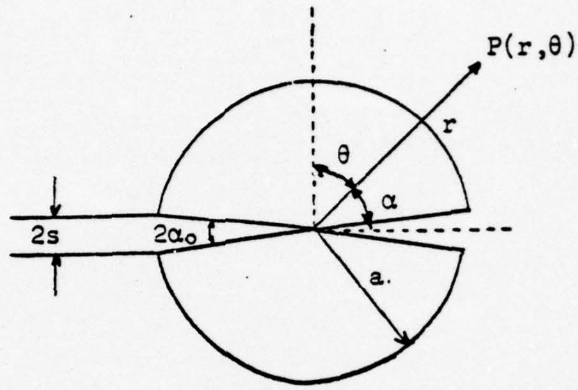


Fig. 8. Geometry of the spherical antenna.

The boundary value problem associated with the radiation from the antenna is the solution of steady-state, source-free Maxwell equations subject to

(i) radiation condition at infinity,

(ii) symmetry condition relative to the ϕ-coordinate,

(iii) boundary condition at $r = a$.

The solution of this problem can be obtained following the same procedure of Ch. I, 3. In particular, the magnetic field has only a ϕ-component given by

$$\hat{H}_\phi(r,\theta) = \frac{ikV_0}{\zeta} \sum_{1}^{\infty}{}_n \frac{4n-1}{2(2n-1)} \frac{h_{2n-1}(kr)}{(2n-1)h_{2n-1}(ka) - ka\, h_{2n-2}(ka)} P^1_{2n-1}(\eta)\, V_{2n-1}(\eta_0) \quad (10.1)$$

where $h_n(x)$ is the spherical Hankel function of the second kind, $k = \omega\sqrt{\varepsilon\mu}$, $\zeta = \sqrt{\mu/\varepsilon}$, $P^1_n(x)$ the associated Legendre polynomial of the first kind and $V_{2n-1}(\eta_0)$, the voltage excitation for mode of order 2n-1, is given by

$$V_{2n-1}(\eta_0) = -\frac{1}{2nV_0}\int_{-\eta_0}^{\eta_0} \frac{\partial\hat{V}}{\partial\eta}\sqrt{1-\eta^2}\, P_{2n-1}^1(\eta)\, d\eta; \tag{10.2}$$

this is a definition slightly different from that of Ch. I, eq. (3.11). Note that for a (spatially) δ-gap excitation $V_n = V_n(0)$ and simplifies upon use of (2.23) as

$$V_{2n-1} = P_{2n}(0). \tag{10.3}$$

The other field components, i.e., E_θ and E_r immediately follow from Maxwell's equations:

$$\hat{E}_r = \frac{\zeta}{ikr\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\hat{H}_\phi) \quad , \quad \hat{E}_\theta = -\frac{\zeta}{ikr}\frac{\partial}{\partial r}(r\hat{H}_\phi). \tag{10.4}$$

The expression of the field in time-domain is obtained, in principle, by inverse Fourier (or Laplace) transform of (10.1 and 10.4) once the Fourier (or Laplace) transform of the applied voltage has been chosen.

Assume that the external medium is free-space and that the applied voltage changes in time as $V_0U(t)$, i.e., as a Heanside step function. Then, the general term of the series expansion (10.1) that must be inverted is:

$$\frac{h_{2n-1}(\omega r/c)}{(2n-1)h_{2n-1}(\omega a/c) - (\omega a/c)h_{2n-2}(\omega a/c)} \tag{10.5}$$

where $c = 1/\sqrt{\varepsilon_0\mu_0}$.

Firstly, consider the term $n = 1$. Use of (2.25) transforms (10.5) into

$$\frac{a}{r}\;\frac{i\omega a/c + a/r}{i\omega a/c + 1 - (\omega a/c)^2}\exp\left(-i\omega\,\frac{r-a}{c}\right) \tag{10.6}$$

which exhibits two poies in the complex frequency plane $p = i\omega$ at

$$p_1 = -\frac{c}{2a} \pm i\,\frac{\sqrt{3}c}{2a} = -\alpha_1 \pm i\omega_1 \tag{10.7}$$

The inverse-Laplace transform contour of integration can be closed in the left-hand half plane for $t > \frac{r-a}{c}$ and the integral easily computed using the residue theorem. Hence

$$H_{\phi_1}(t) = \frac{\sqrt{3}\,V_0}{2\zeta r} \sin\theta \exp(-\alpha_1 t^*)[\cos(\omega_1 t^* + \frac{\pi}{6}) + \frac{a}{2r} \sin \omega_1 t^*],$$

$$t^* = t - \frac{r-a}{c} > 0 \qquad (10.8)$$

At large distances from the antenna $a/r \to 0$, the electric field has only the $E_{\theta 1} = \zeta H_{\phi 1}$ component and the radiated field is a *damped oscillation* with period $4\pi a/c$ and time attenuation constant $2a/c$.

Let us now consider the general term (10.5). Use of (2.25) clearly shows that its behavior for $|p| \to \infty$ is of the type $\exp[p(r-a)/c]/p$. Accordingly, the inverse-Laplace contour of integration can be closed in the left-hand half plane for $t > (r-a)/c$ and the inverse-Laplace transform computed as $2\pi i$ times the sum of the residues at the poles of the integrand. These, in turn, occur at complex frequencies p_s given by

$$(2n-1)h_{2n-1}(-ip_s a/c) + i(p_s a/c)h_{2n-2}(-ip_s a/c) = 0 \qquad (10.9)$$

Note that the (2n-1) order pole of $h_{2n-1}(-ipr/c)$ at $p = 0$ cancels with the analogous same order pole of the denominator.

Use of (2.25) shows eq. (10.9) to have 2n zeros; each residue can be easily computed upon use of (2.26), and is given by:

$$- \frac{p_s \exp(p_s t)}{2n(2n-1) + (p_s a/c)^2} \, \frac{h_{2n-1}(-ip_s r/c)}{h_{2n-1}(-ip_s a/c)} U(t^*), \qquad (10.10)$$

where

$$t^* = t - \frac{r-a}{c} \qquad (10.11)$$

is the *retarded time*.

Relation (10.10) clearly shows that the dominant poles are those closest to the imaginary axis. On the other hand, the analysis of Ch. I, 6 showed that these poles are approximately located at

$$p_{2n-1}a/c = -\ 0.4(2n-1)^{2/3} \pm i(2n-1) \tag{10.12}$$

for n large. Accordingly, at (retarded) observation times *large* compared to $(0.4)(2n-1)^{-2/3}$ a/c, the first few modes of the series (10.1) are already a good representation for the radiated field. Then, the conclusion is reached that the *late-time behavior* of the field is properly described by the inverse-Laplace transform of the first, or first few terms, of the series (10.1).

Let us now study the *early-time behavior* of the field. For this purpose, it is convenient to rearrange the general term (10.5) to be inverted under the product of two functions, hence

$$\frac{h_{2n-1}(\omega a/c)}{(2n-1)h_{2n-1}(\omega a/c) - (\omega a/c)h_{2n-2}(\omega a/c)} \cdot \frac{h_{2n-1}(\omega r/c)}{h_{2n-1}(\omega a/c)} \tag{10.13}$$

The inverse-Laplace transform of the first term can be immediately obtained by using previous results as

$$-\frac{p_s \exp(p_s t)}{2n(2n-1) + (p_s a/c)^2}\, U(t) \tag{10.14}$$

Analogously, the inverse-Laplace transform of the second term is readily seen to be equal to

$$\frac{a}{r}\,\delta(t^*) - i\,\frac{c}{a}\,\sum_q \frac{h_{2n-1}(-ip_q r/c)}{h_{2n}(-ip_q a/c)} \exp(p_q t)\, U(t^*), \tag{10.15}$$

$h_{2n-1}(-ip_q a/c) = 0$, upon use of (2.25-2.26). The inverse-Laplace transform of (10.13) is now obtained by convolution of (10.14 and 10.15). However, at *small*

observation times only the convolution with the $\delta(t^*)$ function will be important, so that the field radiated by the antenna will be given by

$$H_\phi = - \frac{V_0}{\zeta r} \sum_{1}^{\infty}{}_n \frac{4n-1}{2(2n-1)} P^1_{2n-1}(\eta)\, V_{2n-1}(\eta_0) \cdot \sum_{1}^{2n}{}_s \frac{(p_s a/c)\exp(p_s t^*)}{2n(2n-1) + (p_s a/c)^2}\,, \qquad t^* \to 0^+ \tag{10.16}$$

An idea of the convergence properties of the series (10.16) is obtained by considering only dominant poles ($s = 2n-1$), the field in the equatorial plane ($\eta = 0$), and by using the asymptotic expression of the series term for $n >> 1$. Hence

$$H_\phi \simeq \frac{5V_0}{\pi\zeta r} \mathrm{Re}\left[\sum_{1}^{\infty}{}_n \frac{\exp[i(2n-1)ct^*/a - 0.4(2n-1)^{2/3}\, ct^*/a]}{(2n-1)^{2/3}} \right] \tag{10.17}$$

It is readily seen that the convergence of the series (10.17) is very poor indeed when ct^*/a is very small, and that the series diverges for $t^* = 0$.

Considering only the dominant term in the exponent and upon use of (2.27), we have approximately

$$H_\phi \simeq \frac{5.8\, V_0}{\pi\zeta r} \frac{1}{(ct^*/a)^{1/3}} \tag{10.18}$$

Although largely approximated, result (10.18) allows us to conclude that, at *small observation times*, the inverse-Laplace transform of the series expansion (10.1) is very poorly convergent and the field is inversely proportional to a *fractional power of the retarded time* for a step time excitation.

On the other hand, the early-time behavior of the field can be computed by using results of Sec. 5. As a matter of fact, at time $t = 0$ a magnetic ring current

$$I_m(t) = V_0 U(t) \tag{10.19}$$

is applied at the equatorial plane of the antenna. For $t^* \simeq 0$ the transient radiated field should *not* depend on the particular antenna excited by the magnetic ring current. Accordingly the transient field should be coincident with that of a magnetic loop on a metal surface. By modifying results of Sec. 5 for $\theta = 90^0$ and by using duality we get, for $ct^*/a \to 0$,

$$H_\phi = \frac{V_0}{\sqrt{2}\pi\zeta r} \frac{1}{\sqrt{ct^*/a}} , \tag{10.20}$$

which exhibits behavior of the type (10.18). This proves that the early time radiation from the antenna essentially comes out from the gap. Using the results of Sec. 5 for any θ we get

$$\left\{ \begin{array}{ll} E_\theta = \dfrac{V_0}{\sqrt{2}\pi r} \dfrac{1}{\sqrt{ct^* \sin\theta/a}} , \quad ct^*/a \sin\theta << 1 & (10.21) \\ H_\phi = \dfrac{E_\theta}{\zeta} , \quad t^* = t - \dfrac{r-a \sin\theta}{c} & (10.22) \end{array} \right.$$

When a/c is negligible compared to the time resolution of the measuring device, the procedure developed under Sec. 5 transforms (10.21-10.22) in

$$\left\{ \begin{array}{ll} E_\theta = \dfrac{V_0 a}{\pi r c} \sin\theta \ \delta(t^*) & (10.23) \\ H_\phi = \dfrac{E_\theta}{\zeta} & (10.24) \end{array} \right.$$

since only one half of the magnetic loop contributes to the radiated field, the second half being screened by the metal body of the antenna. Accordingly, the early-time transient field is proportional to the time derivative of the applied voltage. Incidently, results (10.21-10.22) are in complete agreement with the early-time behavior of the field radiated by an infinitely long cylindrical antenna.

CHAPTER III

TRANSFORM PROPERTIES OF RADIATED FIELDS

Fourier, Fourier, how many errors are made in your name!

1. The problem

In elementary antenna theory it is well known that the far field radiated from a wire antenna is just the *Fourier transform* - within a proportionality factor - of the current distribution. This is a rather important property from the applications point of view. It is helpful from the *analysis* point of view - calculate the field radiated from a given antenna current - and it is crucial from the *synthesis* point of view - from a given radiated field, calculate the current distribution.

The above properties of radiated fields can be stated in an elegant way for the case of *aperture antennas*. It is assumed that an aperture exists in a metal body and that a *known* field distribution is produced across it. When the surface of the metal body coincides with a coordinate surface of a number of coordinate systems - e.g., an infinite plane, a circular cylinder, a circular cone, a sphere - it is possible to express the field associated with the aperture as the superposition of elementary fields constituting a complete basis. In the simplest case - an aperture in an infinite metal plane - the elementary fields are plane waves, so that the *plane-wave expansion* of the radiated field is considered. It can be shown that this plane-wave expansion allows for the *directivity diagram*, the *directivity* and the *superdirectivity ratio* to be easily computed.

A further transform property of the field is obtained by noting that the aperture is space limited. Accordingly, it can be shown that *Hilbert-transform*

relations link real and imaginary parts of the radiated field. This is a rather important property from the practical point of view, since it reveals physical constraints to which the radiated field must comply. Knowledge of this constraint allows for realistic synthesis procedures to be developed, for measuring procedures to be simplified, for numerical methods and computations to be verified, and so on.

2. Relevant formulas and pertinent expansions

Transformations of "del" operator

$\nabla \cdot [\underline{A} \exp(-i\,\underline{k} \cdot \underline{r})] = \underline{A} \cdot \nabla \exp(-i\,\underline{k} \cdot \underline{r}) = -i\,\underline{A} \cdot \underline{k} \exp(-i\,\underline{k} \cdot \underline{r}) \nabla \cdot \underline{r} =$

$= -i\,\underline{k} \cdot \underline{A} \exp(-i\,\underline{k} \cdot \underline{r})$. Then, formally

$$\nabla \cdot \rightarrow -i\,\underline{k}. \tag{2.1}$$

$\nabla \times [\underline{A} \exp(-i\,\underline{k} \cdot \underline{r})] = -\,\underline{A} \times \nabla \exp(-i\,\underline{k} \cdot \underline{r}) = -i\,\underline{k} \times \underline{A} \exp(-i\,\underline{k} \cdot \underline{r})$

Then, formally

$$\nabla \times \rightarrow -i\,\underline{k}\, \times \tag{2.2}$$

Bessel functions

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^\infty \exp(in\xi + i\,x \cos\xi)\,d\xi. \tag{2.3}$$

Asymptotic evaluation of integrals

Let $I(\Omega) = \int_G f(z) \exp[-i\Omega q(z)]dz$ with $q'(z_s) = 0$, $q''(z_s) \neq 0$; z_s is called a saddle point. Then, in the neighborhood of $z = z_s$,

$$q(z) = q(z_s) + \frac{q''(z_s)}{2}(z - z_s)^2 + \ldots .$$

Letting $\frac{q''(z_s)}{2}(z - z_s)^2 = s^2$ (stationary phase path), for $\Omega \rightarrow \infty$

$$I(\Omega) = f(z_s)\ \exp[-i\Omega q(z_s)]\ \frac{dz}{ds}\Bigg|_{s=0} \int_{-\infty}^{+\infty} \exp(-is^2)ds =$$

$$= \sqrt{\frac{-2\pi i}{\Omega q''(z_s)}}\ f(z_s)\ \exp[-i\Omega q(z_s)] \tag{2.4}$$

provided that the original integration contour G can be deformed into the stationary phase path and the saddle point is sufficiently far from singularities of the integrand.

Similarly, in two-dimensions

$$\int du \int dv\ f(u,v)\ \exp[-i\Omega q(u,v)] \simeq$$

$$\simeq -\frac{2\pi i}{\Omega\sqrt{\det \underline{\underline{H}}}}\ f(u_s,v_s)\ \exp[-i\Omega q(u_s,v_s)] \tag{2.5}$$

where

$$\underline{\underline{H}} = \begin{Vmatrix} \dfrac{\partial^2 q}{\partial u^2} & \dfrac{\partial^2 q}{\partial u \partial v} \\ \\ \dfrac{\partial^2 q}{\partial v \partial u} & \dfrac{\partial^2 q}{\partial v^2} \end{Vmatrix} \tag{2.6}$$

Parseval's theorem in two-dimensions

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\ f(x,y)\ g^*(x,y) = (2\pi)^2 \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\ \hat{f}(u,v)\ \hat{g}^*(u,v) \tag{2.7}$$

where $\hat{f}(u,v)$ and $\hat{g}(u,v)$ are the inverse Fourier transforms of $f(x,y)$ and $g(x,y)$, respectively.

Integrals

$$\int \cos\left(\frac{\pi\alpha^2}{2b^2}y^2 - vy\right)dy = \frac{b}{\alpha}\left[\cos\frac{v^2b^2}{2\pi\alpha^2}\ C\left(\frac{\alpha y}{b} - \frac{vb}{\pi\alpha}\right) + \sin\frac{v^2b^2}{2\pi\alpha^2}\ S\left(\frac{\alpha y}{b} - \frac{vb}{\pi\alpha}\right)\right] \tag{2.8}$$

$$\int \sin\left(\frac{\pi\alpha^2}{2b^2} y^2 - vy\right) dy = \frac{b}{\alpha}\left[\cos \frac{vb^2}{2\pi\alpha^2} S\left(\frac{\alpha y}{b} - \frac{vb}{\pi\alpha}\right) - \sin \frac{vb^2}{2\pi\alpha^2} C\left(\frac{\alpha y}{b} - \frac{vb}{\pi\alpha}\right)\right] \tag{2.9}$$

where $C(x)$, $S(x)$ are the Fresnel integrals.

Fresnel integrals and auxiliary functions

$$C(t) = \frac{1}{2} + f(t) \sin\left(\frac{\pi t^2}{2}\right) - g(t) \cos\left(\frac{\pi t^2}{2}\right) \tag{2.10}$$

$$S(t) = \frac{1}{2} - f(t) \cos\left(\frac{\pi t^2}{2}\right) - g(t) \sin\left(\frac{\pi t^2}{2}\right) \tag{2.11}$$

where $f(t)$ and $g(t)$ are the so-called *auxiliary functions*. For $t \to 0$:

$$f(t) = \frac{1}{2} - \frac{\pi}{4} t^2 + \ldots ; \quad g(t) = \frac{1}{2} - t + \ldots \tag{2.12}$$

For $t \to \infty$:

$$f(t) \simeq \frac{1}{\pi t} \quad ; \quad g(t) \simeq \frac{1}{\pi t^3} \tag{2.13}$$

$$f'(t) = -\pi t\, g(t) \quad ; \quad g'(t) = \pi t\, f(t) - 1 \tag{2.14}$$

3. Plane wave expansion of electromagnetic waves

Let us consider the electromagnetic field $(\underline{E}, \underline{H})$, solution of the source-free Maxwell equations

$$\nabla \times \underline{E} = -i\omega\mu \underline{H} \tag{3.1}$$

$$\nabla \times \underline{H} = i\omega\varepsilon \underline{E} \tag{3.2}$$

$$\nabla \cdot \underline{E} = 0 \tag{3.3}$$

$$\nabla \cdot \underline{H} = 0 \tag{3.4}$$

The environment is assumed homogeneous and isotropic, and steady-state conditions are postulated.

Assume now the spatial dependence of the fields to be of the type $\exp(-i\,\underline{k}\cdot\underline{r})$. Then, upon substitution into (3.1 through 3.4), the result

$$\underline{k}\times\underline{\hat{E}} = \omega\mu\underline{\hat{H}} \tag{3.5}$$

$$\underline{\hat{H}}\times\underline{k} = \omega\varepsilon\underline{\hat{E}} \tag{3.6}$$

$$\underline{k}\cdot\underline{\hat{E}} = 0 \tag{3.7}$$

$$\underline{k}\cdot\underline{\hat{H}} = 0 \tag{3.8}$$

is obtained, where $\hat{E}$ and $\hat{H}$ are spatially constant vectors. Note that use has been made of (2.1-2.2).

Let us now cross-multiply eq. (3.5) by $\underline{k}$ from the right, and substitute from (3.6). We get

$$\underline{k}\times\underline{\hat{E}}\times\underline{k} - \omega^2\varepsilon\mu\underline{\hat{E}} = 0 \tag{3.9}$$

Eq. (3.7) shows that $\underline{\hat{E}}$ and $\underline{k}$ are orthogonal. Then, letting $\omega^2\varepsilon\mu = \beta^2$, where β is the free-space propagation constant,

$$(k^2 - \beta^2)\underline{\hat{E}} = 0 \tag{3.10}$$

Accordingly, the absolute value of $\underline{k}$ is determined, and equals β. Let u,v,w be the Cartesian components of $\underline{k}$:

$$\underline{k} = u\underline{\hat{x}} + v\underline{\hat{y}} + w\underline{\hat{z}}\ ; \tag{3.11}$$

it follows from $k = \beta$ that

$$w = \sqrt{\beta^2 - (u^2 + v^2)} \tag{3.12}$$

Accordingly, the (u,v) plane can be divided into two regions separated by a circle centered at the origin and of radius β. For the inner region, which is called the *visible region*, $u^2 + v^2 < \beta^2$ and w is real. The square-root in (3.12) is taken with the positive sign, so that propagation along the

positive direction of the z-axis is assured. The propagation vector $\underline{k}$ is determined by direction cosines

$$\cos\phi_1 = \frac{u}{\beta} \;;\; \cos\phi_2 = \frac{v}{\beta} \;;\; \cos\theta = \frac{w}{\beta} \tag{3.13}$$

which correspond to real angles (see Fig. 1).

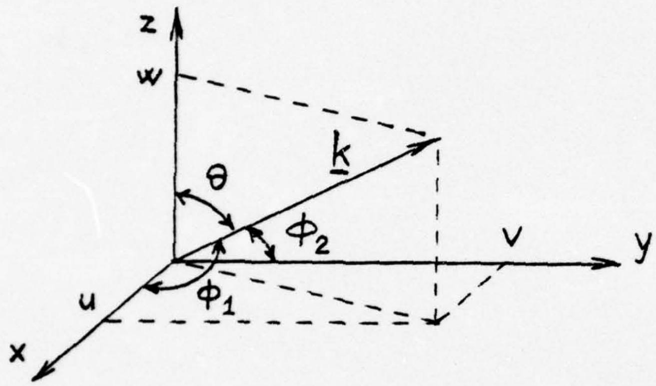


Fig. 1 - Relevant to the propagation vector $\underline{k}$

For $u^2 + v^2 = \beta^2$, $\theta = 90°$ and the propagation vector $\underline{k}$ is parallel to the $z = 0$ plane.

For $u^2 + v^2 > \beta^2$, which defines the *invisible region*, w is pure imaginary, and we take

$$w = -i\,|w| = -i\sqrt{(u^2 + v^2) - \beta^2} \quad , \quad u^2 + v^2 > \beta^2 \tag{3.14}$$

so that damping of the wave along the positive direction of the z-axis is assured. Eq. (3.14) corresponds to a complex value of θ angle, $\theta = 90° + i\,\bar{\theta}$, where

$$\cos\theta = -\,i\,\sinh\bar{\theta} = -\,i\,\frac{|w|}{\beta} \tag{3.15}$$

Real values of w correspond to the usual plane waves; imaginary values to surface waves linked to the plane $z = 0$ and exponentially attenuated for $z > 0$.

As $\underline{k}$ components are not all independent, also $\hat{\underline{E}},\hat{\underline{H}}$ components are mutually related. In particular

$$\zeta\,\hat{\underline{H}} = \frac{\underline{k}\times\hat{\underline{E}}}{\beta} \tag{3.16}$$

from (3.5), where $\zeta = \sqrt{\mu/\varepsilon}$, and

$$\hat{E}_z = -\,\frac{u\,E_x + v\,E_y}{w} \tag{3.17}$$

from (3.7).

Under rather general conditions, an electromagnetic field $(\underline{E},\underline{H})$ can be expanded in a set of plane waves. Hence, for $z > 0$

$$\underline{E}(x,y,z) = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\ \hat{\underline{E}}(u,v)\ \exp(-i\,w\,z)\ \exp(-i\,u\,x - i\,v\,y) \tag{3.18}$$

Note that $\hat{\underline{E}}$, which is (u,v)-dependent, is a density of electric field per square-wavenumber, i.e., it is measured in volt•m. Obviously, the formal superposition (3.18) is of practical use if the vector function $\hat{\underline{E}}$ can be determined once the sources of $\underline{E}$ are given.

4. Radiation from an aperture. Spectrum calculation

Let us now cross-multiply (3.18) by $\hat{\underline{z}}$ from both left and right, and specify the result for $z = 0$. We get

$$\hat{\underline{z}}\times\underline{E}(x,y,0)\times\hat{\underline{z}} = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\ \hat{\underline{z}}\times\hat{\underline{E}}(u,v)\times\hat{\underline{z}}\ \exp(-i\,u\,x - i\,v\,y) \tag{4.1}$$

Note that $\hat{\underline{z}}\times\underline{E}(x,y,0)\times\hat{\underline{z}} = \underline{E}_t^a(x,y)$ is the tangential component of the electric field in the plane $z = 0$, i.e., the applied tangential electric field across the aperture. Letting $\hat{z}\times\hat{\underline{E}}(u,v)\times\hat{\underline{z}} = \hat{\underline{E}}_t(u,v)$, eq. (4.1) can be rewritten as

$$\underline{E}_t^a(x,y) = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\ \hat{\underline{E}}_t(u,v)\ \exp(-i\,u\,x - i\,v\,y) \tag{4.2}$$

Eq. (4.2) shows the tangential electric field across the aperture to be the double Fourier-transform of the (x,y) components of the plane-wave expansion of the electric field associated with the aperture. Accordingly, by inverting (4.2), the result

$$\hat{\underline{E}}_t(u,v) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\ \underline{E}_t^a(x,y)\ \exp(i\,u\,x + i\,v\,y) \tag{4.3}$$

is obtained. Note that the integration in (4.3) can be limited to the aperture itself, since $\underline{E}_t(x,y) = 0$ in the remaining part of the plane $z = 0$. The vector $\hat{\underline{E}}_t(u,v)$ is called the *spectrum* or *spatial spectral components*, associated with the aperture; analogously, $\underline{k}$ is called the spatial frequency associated with the aperture. The third component of the complete plane-wave expansion can easily be obtained via (3.17).

If $\underline{E}_t^a(x,y)$ can be factorized into two functions of a single variable, i.e.,

$$\underline{E}_t^a(x,y) = -\nabla_t\ f_1(x)\ f_2(y), \tag{4.4}$$

(as it happens in rectangular waveguides for TM modes), substitution into (4.3) and integration by parts leads to the result

$$\hat{\underline{E}}_t(u,v) = i\ F_1(u)\ F_2(v)\underline{k}_t \tag{4.5}$$

where

$$F_1(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_1(x)\ \exp(i\,u\,x)dx \tag{4.6}$$

$$F_2(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_2(y)\ \exp(i\,v\,y)dy \tag{4.7}$$

are the inverse Fourier transforms of $f_1(x)$, $f_2(y)$ respectively, and $\underline{k}_t = u\hat{\underline{x}} + v\hat{\underline{y}}$ is the transversal component of the wavenumber vector.

Let now (ρ,ϕ) be polar coordinates in the $z=0$ plane, and let $\underline{k}_t = k_t \cos\psi\,\hat{x} + k_t \sin\psi\,\hat{y}$. Should the factorization

$$\hat{\underline{E}}_t^a(\rho,\phi) = -\nabla_t\, f(\rho)\, \exp(in\phi) \tag{4.8}$$

be possible (as it happens in circular waveguides for TM modes), we analogously get

$$\underline{E}_t(u,v) = \frac{i\,\underline{k}_t}{(2\pi)^2} \int_0^\infty f(\rho)\rho\, d\rho \int_\psi^{\psi+2\pi} \exp(in\phi)\, \exp(i\,\underline{k}_t \cdot \underline{\rho})\, d\phi =$$

$$= i\,\underline{k}_t\, \frac{\exp(in\psi)}{(2\pi)^2} \int_0^\infty f(\rho)\rho\, d\rho \int_0^{2\pi} \exp(in\xi + i\, k_t \rho \cos\xi)\, d\xi \tag{4.9}$$

Then, by using (2.3), the result

$$\hat{\underline{E}}_t(u,v) = i\,\underline{k}_t\, i^n \exp(in\psi)\, \frac{1}{2\pi} \int_0^\infty f(\rho)\, J_n(k_t\rho)\rho\, d\rho \tag{4.10}$$

is obtained. Accordingly, $\hat{\underline{E}}_t$ is factorized in terms of two functions of ψ and ρ respectively, the latter being the inverse Fourier-Bessel transform of the aperture field function distribution $f(\rho)$.

5. Radiation from an aperture continued. Far-field calculation

Let us investigate how the integral (3.18) simplifies when the electric field is evaluated at large distances $\underline{r}$ from the aperture. A particular spatial direction is determined by its directional cosines (ξ,η,σ):

$$\underline{r} = r(\xi\hat{\underline{x}} + \eta\hat{\underline{y}} + \sigma\hat{\underline{z}}) \tag{5.1}$$

where $\xi^2 + \eta^2 + \sigma^2 = 1$. Along this particular direction, at a large distance from the aperture, the electric field $\underline{E}(\underline{r})$ may be written as the superposition of elementary plane wave fields according to (3.18). However, at *large distances*

from the aperture, the field is a locally plane wave. Accordingly, we can reasonably surmise that the field will be *mainly* set up by those plane wave constituents whose propagation vectors are contained within a small cone whose axis is the considered direction.

In order to make this intuitive reasoning mathematically sound, let us compute the value of the argument of the exponential in (3.18), namely $g(u,v) = ux + vy + wz$, when the propagation vector $\underline{k}$ is oriented in the neighborhood of the considered direction (ξ,η,σ). This, in turn, requires the first two cosine directions of $\underline{k}$ to be in the neighborhood of u_0,v_0, where

$$u_0 = \beta\xi \quad ; \quad v_0 = \beta\eta \tag{5.2}$$

Accordingly, we have

$$g(u_0,v_0) = \beta r(\xi^2 + \eta^2 + \sigma^2) = \beta r \tag{5.3}$$

$$\left.\frac{\partial g}{\partial u}\right|_{u_0,v_0} = \beta r\left(\xi - \frac{\sigma\xi}{\sqrt{1 - \xi^2 - \eta^2}}\right) = 0 \tag{5.4}$$

$$\left.\frac{\partial g}{\partial v}\right|_{u_0,v_0} = \beta r\left(\eta - \frac{\sigma\eta}{\sqrt{1 - \xi^2 - \eta^2}}\right) = 0 \tag{5.5}$$

It follows that $g(u,v)$ is *stationary* in the neighborhood of (u_0,v_0). The main contribution to the integral (3.18) then arises from an integration area centered at (u_0,v_0) and vanishingly small as $\beta r \to \infty$. As a matter of fact, outside this area the phase of the integrand is a rapidly varying function of (u,v). Accordingly, as $\beta r \to \infty$

$$\underline{E}(r,\theta,\phi) \simeq \hat{\underline{E}}(u_0,v_0)\ \exp(-i\,\beta r)\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\ \exp[-i(\underline{k}\cdot\underline{r} - \beta r)] =$$

$$= 2\pi i\ \frac{\beta\ \exp(-i\,\beta r)}{r}\ \cos\theta\ \hat{\underline{E}}(u_0,v_0) \tag{5.6}$$

Note that use has been made of (2.5) for evaluating the integral.

Relation (5.6) states that the *far-field radiation diagram* of the aperture is determined - aside from angular factor $\cos\theta$ - by its plane-wave associated expansion. For a given (spherical coordinates) direction (θ,ϕ) the values (u_0,v_0) appearing in (5.6) are given by

$$u_0 = \beta \sin\theta \cos\phi \quad ; \quad v_0 = \beta \sin\theta \sin\phi \tag{5.7}$$

6. Example. The rectangular aperture

Let us now consider a rectangular aperture of dimensions (2a,2b). The aperture lies in the $z = 0$ plane. A system of rectangular Cartesian coordinates is introduced, centered at the center of the aperture, with the x-axis parallel to the side of length 2a and the y-axis parallel to the side 2b. The aperture is illuminated by an electromagnetic field whose tangential electric field component is given by

$$\underline{E}^a = E_0 \hat{y} \cos\left(\frac{\pi}{2a}\, x\right) \tag{6.1}$$

The tangential component $\hat{\underline{\underline{E}}}_t$ of the spectrum assoicated with the aperture is readily computed as

$$\hat{\underline{\underline{E}}}_t(u,v) = \hat{E}_y\, \hat{y} = \frac{E_0 \hat{y}}{(2\pi)^2} \int_{-b}^{b} \exp(i\,v\,y)\,dy \int_{-a}^{a} \cos\left(\frac{\pi}{2a}\, x\right) \exp(i\,u\,x)\,dx =$$

$$= E_0\, \hat{y}\, \frac{2ab}{\pi^3}\, \frac{\sin vb}{vb}\, \frac{\cos u a}{1 - \left(\frac{2\,u\,a}{\pi}\right)^2} \tag{6.2}$$

the z-component of the spectrum follows from (3.17), hence

$$\hat{E}_z = - \frac{v\, \hat{E}_y}{\sqrt{\beta^2 - (u^2 + v^2)}} \tag{6.3}$$

Let us now study the radiation diagram in two principal planes.

The E-plane is defined by $\phi = \pm 90°$. For this plane we have, from (5.7), $u_0 = 0$, $v_0 = \pm \beta \sin\theta$. Then, from (6.3)

$$\hat{E}_z = \pm \tan\theta \, \hat{E}_y \tag{6.4}$$

Accordingly, the far-field will have components

$$\begin{cases} E_\theta = - i E_0 \dfrac{8\ ab}{\pi \lambda r} \exp(-i\beta r) \left[\dfrac{\sin(\beta\, b \sin\theta)}{\beta\, b \sin\theta}\right] & (6.5) \\ H_\phi = \dfrac{E_\theta}{\xi} & (6.6) \end{cases}$$

where use has been made of (5.6).

The bracketed term in (6.5) is clearly the radiation diagram in the E-plane. The first radiation null occurs at an angle θ_0 given by

$$\beta\, b \sin\theta_0 = \pi. \tag{6.7}$$

For *large apertures* ($\beta b \gg 1$), θ_0 is small and the width of the first lobe is given by

$$\Delta\theta_0 \simeq \frac{\lambda}{b} \tag{6.8}$$

The first lateral lobe occurs at

$$\beta b \sin\theta = \frac{3\pi}{2} \tag{6.9}$$

Its amplitude is equal to $2/3\pi$, i.e., lateral lobes are more than 14db below the principal lobe.

The H-plane is defined by $\phi = 0, 180°$. For this plane we have, from (5.7), $u_0 = \pm \beta \sin\theta$, $v_0 = 0$. Then, from (6.3), $\hat{E}_z = 0$ and the far-field will have components

$$\begin{cases} E_\phi = -\,i\,E_0 \dfrac{8\,ab}{\pi\,\lambda\,r} \cos\theta \, \exp(-i\beta r) \left[\dfrac{\cos(\beta\,a\sin\theta)}{1 - \left(\dfrac{2\beta\,a\sin\theta}{\pi}\right)^2} \right] & (6.10) \\ H_\theta = -\,\dfrac{E_\phi}{\xi} & (6.11) \end{cases}$$

where use has been made of (5.6).

The bracketed term in (6.10) times $\cos\theta$ is clearly the radiation diagram in the H-plane. The first radiation null occurs at an angle θ_0 given by

$$\beta\,a\sin\theta_0 = \frac{3\pi}{2} \tag{6.12}$$

Note that, at $\beta\,a\sin\theta = \pi/2$, the radiation diagram is not zero, taking instead the value $\pi/4$. For large apertures we have

$$\Delta\theta_0 \simeq \frac{3\lambda}{2a} \tag{6.13}$$

The first lateral lobe occurs at

$$\beta\,a\sin\theta = 2\pi \tag{6.14}$$

Its amplitude is equal to 1/15, i.e., lateral lobes are more than 25 db below the principal lobe.

Previous discussion highlights the fact that an *amplitude tapering* of the field in the aperture reduces the amplitude of lateral lobes at the expense of an increase in the width of the principal lobe.

7. Example continued. The effect of phase tapering

The effect of phase tapering in the aperture is now examined. To this end, the tangential component of the electric field in the aperture is assumed equal to

$$\underline{E}^a = E_0\,\hat{y}\,\sin\left(\frac{\pi}{2a}\,x\right)\exp\left[i\,\frac{\pi\alpha^2}{2}\left(1 - \frac{y^2}{b^2}\right)\right] \tag{7.1}$$

Comparison of (7.1) with (6.1) shows that the u-dependence of $\hat{E}_\theta(u,v)$ is the same as in (6.2), while the v-dependence is given by

$$Q(v,\alpha) = \exp(i\,\frac{\pi}{2}\,\alpha^2)\int_{-b}^{b}\exp\left(i\,v\,y - i\,\frac{\pi}{2}\,\alpha^2\,\frac{y^2}{b^2}\right)dy =$$

$$= \frac{b}{\alpha}\left\{\left[g\left(\frac{vb}{\pi\alpha} - \alpha\right) - i\,f\left(\frac{vb}{\pi\alpha} - \alpha\right)\right]\exp(i\,v\,b)\right.$$

$$\left. - \left[g\left(\frac{vb}{\pi\alpha} + \alpha\right) - i\,f\left(\frac{vb}{\pi\alpha} + \alpha\right)\right]\exp(-i\,v\,b)\right\} \qquad (7.2)$$

where the functions f and g are defined by (2.10-2.11). If the further assumption that α is small is made, the functions f and g can be expanded in the neighborhood of $vb/\pi\alpha$ and only the first two terms of the expansion retained. Then, upon use of (2.14), the result

$$Q(v,\alpha) = 2b\left\{\frac{\sin v\,b}{vb}\left[\frac{vb}{\alpha}\,f\left(\frac{vb}{\pi\alpha}\right) + i\,\frac{vb}{\alpha}\,g\left(\frac{vb}{\pi\alpha}\right)\right]\right.$$

$$\left. + \cos v\,b\left[-\,\frac{vb}{\alpha}\,f\left(\frac{vb}{\pi\alpha}\right) - i\,\frac{vb}{\alpha}\,g\left(\frac{vb}{\pi\alpha}\right) + 1\right]\right\} \qquad (7.3)$$

is obtained.

The effect of the (quadratic) phase tapering in the E-plane is now studied. Two angle ranges are considered. Firstly, assume

$$\frac{\beta\,b\sin\theta}{\pi\alpha} = \frac{2b\sin\theta}{\lambda\alpha} << 1 \qquad (7.4)$$

i.e., angles θ much smaller than α. Then, upon use of (2.12),

$$Q(v,\alpha) \simeq 2b\cos(\beta\,b\sin\theta) \qquad (7.5)$$

Accordingly, the radiation diagram is not appreciably changed with respect to the case of uniform phase illumination.

Conversely, assume

$$\frac{2b\sin\theta}{\lambda\alpha} >> 1, \qquad (7.6)$$

then, using (2.13), expression (7.3) reduces to

$$Q(v,\alpha) = 2b\left[\frac{\sin v\,b}{v\,b} - i\left(\frac{\pi\alpha}{vb}\right)^2 \cos v\,b\right]. \tag{7.7}$$

Accordingly, the radiation nuls are "filled up".

The previous discussion shows that a *quadratic phase tapering* (or quadratic phase error, as is usually called) tends to *smooth* the radiation diagram.

8. Directivity and superdirectivity ratio

The (complex) power associated with the aperture is given by

$$P = \frac{1}{2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\ \underline{E} \times \underline{H}^* \cdot \underline{\hat{z}} = \frac{1}{2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy (\underline{\hat{z}} \times \underline{E}) \cdot \underline{H}^* =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy (\underline{\hat{z}} \times \underline{E}) \cdot (\underline{\hat{z}} \times H^* \times \underline{\hat{z}}) \tag{8.1}$$

By applying Parsevals' theorem and (3.5), eq. (8.1) can be transformed as follows

$$P = \frac{1}{2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy (\hat{z} \times \underline{E}) \times (\hat{z} \times \underline{H})^* \cdot \underline{\hat{z}} =$$

$$= \frac{1}{2\omega\mu}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv (\underline{\hat{z}} \times \underline{\hat{E}}) \times (\underline{\hat{z}} \times \underline{k} \times \underline{\hat{E}})^* \cdot \underline{\hat{z}} \tag{8.2}$$

Relation (8.2) can be recast in a different way by letting $\underline{\hat{E}} = \underline{\hat{E}}_t + \hat{E}_z \underline{\hat{z}}$, $\underline{k} = \underline{k}_t + w\,\hat{z}$, evaluating the vector products and recalling that u and v are real. Then

$$P = \frac{(2\pi)^2}{2\omega\mu}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \left[w^* |\underline{\hat{E}}_t|^2 + w\,|E_z|^2\right] \tag{8.3}$$

Let us now distinguish, in (8.3), the integration in those regions for which w is real (*visible domain*) and imaginary (*invisible domain*), respectively.

Calling C_R and C_I the two domains respectively, eq. (8.3) transforms into

$$P = \frac{(2\pi)^2}{2\xi} \iint\limits_{C_R} \frac{w}{\beta} |\underline{\hat{E}}|^2 \, du\, dv + i \frac{(2\pi)^2}{2\xi} \iint\limits_{C_I} \frac{|w|}{\beta} \left(|\underline{\hat{E}}_t|^2 - |\hat{E}_z|^2 \right) du\, dv \quad (8.4)$$

The first integral which appears in (8.4) is clearly the (real) *radiated (or active) power* from the aperture; the second integral is the *reactive power* associated with the aperture, i.e., $1/2\omega$ times the difference between the time-averaged values of the magnetic and electric type energies stored in the half-space $z \geq 0$. Accordingly, that part of the integral (6.4) containing $|\underline{\hat{E}}_t|^2$ represents the (magnetic type) stored power while that part containing $|\hat{E}_z|^2$ represents the (electric type) stored power. The total *stored power* is given by

$$P_I = \frac{(2\pi)^2}{2\xi} \iint\limits_{C_I} \frac{|w|}{\beta} |\underline{\hat{E}}|^2 \, du\, dv \quad (8.5)$$

The *directivity ratio* of the aperture is defined as the (spatial) density power radiated in the direction of main radiation divided by the (spatial) mean radiated power. Assume the main radiation to take place along $\theta = 0°$, i.e., in the direction of the z-axis. Then, using (5.6 and 8.4), we get

$$D = \frac{4\pi \, \beta^3 \, |\underline{\hat{E}}(0,0)|^2}{\iint\limits_{C_R} w \, |\underline{\hat{E}}(u,v)|^2 \, du\, dv} \quad (8.6)$$

Note that the (spatial) mean radiated power has been computed with reference to the solid angle 4π.

The *superdirectivity ratio* γ of the aperture is defined as the total (radiated plus stored) power associated with the aperture, divided by the radiated power. Using (8.4-8.5) we get

$$\gamma = \frac{\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \ |w| \ |\hat{\underline{E}}(u,v)|^2}{\iint_{C_R} w \ |\hat{\underline{E}}(u,v)|^2 \ du\, dv} \tag{8.7}$$

Assume now the value $\hat{\underline{E}}(0,0)$ to be fixed so that the value of the radiated field in the direction $\theta = 0$ is determined via (5.6). Note that the electric field is normal to z along this particular direction. From (4.3):

$$\hat{\underline{E}}(0,0) = \hat{\underline{E}}_t(0,0) = \frac{1}{(2\pi)^2} \int dx \int dy \ \underline{E}_t^a(x,y) \tag{8.8}$$

so that the (spatial) *mean value* of the applied electric field across the aperture is fixed.

In order to increase the directivity D, the integral which appears in the denominator of (8.6) should be kept as low as possible. This implies that the spectral components of $\hat{\underline{E}}$ in the C_R domain should be kept as low as possible. This result can be accomplished by illuminating the aperture with a tangential field $\underline{E}^a$ which exhibits a (spatial) rapid variation, so that the (spectral) low-frequency components of its associated spectrum will be lowered. Note that the (spatial) mean value of $\underline{E}^a$ should be different from zero.

This type of aperture illumination has the consequence that the (spectral) high-frequency components of the associated spectrum will be enhanced. Accordingly, the superdirectivity ratio will be increased and, consequently, the stored electric and magnetic power in the neighborhood of the aperture will then be very sensitive to a change in the (temporal) frequency of the applied signal. As a consequence, the transmitted bandwidth will be decreased. In conclusion, the superdirectivity ratio should be chosen according to the bandwidth to be transmitted; this results in an upper limit to the possible obtainable directivity.

9. Hilbert transform properties of the fields

Let f(x) be a square-integrable function such that $f(x) = 0$ for $x < 0$ and let

$$\hat{f}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\ \exp(i\,u\,x) dx \tag{9.1}$$

be its inverse Fourier transform. Let $z = u + i\,u'$ and $\hat{f}(z)$ be the analytic continuation of $\hat{f}(u)$ onto the conplex E-plane. Inverting (9.1) we get

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(u)\ \exp(-i\,u\,x) dx \tag{9.2}$$

The integration contour which appears in (9.2) can be conveniently closed by means of a circle of infinite radius in the half-plane $u' > 0$, for $x < 0$. Then, since $f(x) = 0$ for $x < 0$, it follows that $\hat{f}(z)$ should be singularity-free in the half-plane $u' \geq 0$, (the real axis is included).

Let us now consider the integral

$$\int_C \frac{\hat{f}(z)}{z-u_0} dz \tag{9.3}$$

along the contour C depicted in fig. 2. Since, from (9.1), $\hat{f}(z)$ tends uniformly to zero as $|z| \to \infty$, the integration contour C can be conveniently closed by means of a circle of infinite radius in the half-plane $u' > 0$. However, f(z) is singularity-free in this half-plane. Accordingly, the integration along the above closed contour equals zero, i.e.,

$$⨍_{-\infty}^{+\infty} \frac{\hat{f}(u)}{u-u_0} - \pi\, i\, \hat{f}(u_0) = 0 \tag{9.4}$$

where the integral is to be understood as the principal value and the second term in (9.4) accounts for one-half of the pole contribution at $u = u_0$. Now, if real and imaginary parts of $\hat{f}(u)$ are separated, namely if $\hat{f}(u) = \hat{f}_1(u) + i\,\hat{f}_2(u)$, relation (9.4) splits into

$$\hat{f}_2(u) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\hat{f}_1(u)}{u-u_0}\, du \tag{9.5}$$

$$\hat{f}_1(u) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\hat{f}_2(u)}{u-u_0}\, du \tag{9.6}$$

which are called *Hilbert-transform pair.*

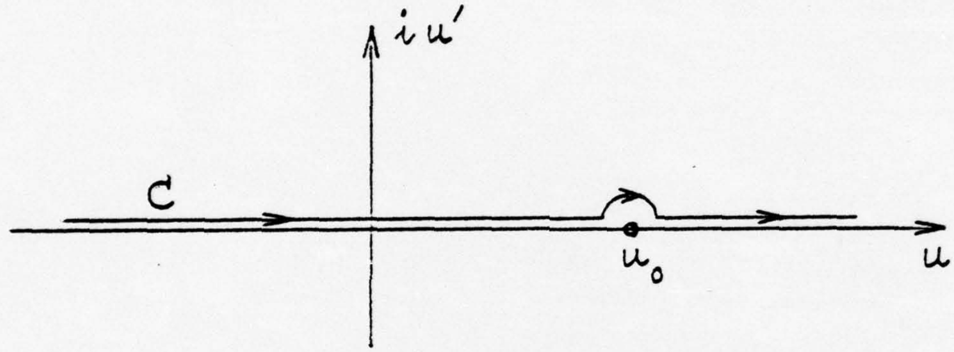


Fig. 2 - Relevant to the Hilbert transform definition

The generalization of (9.4) to a two-variable function $\hat{f}(u,v)$, double inverse Fourier transform of $f(x,y)$ such that $f(x,y) = 0$ for any $x < 0$ and any $y < 0$, is the following

$$\int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \frac{\hat{f}(u,v)}{(u-u_0)(v-v_0)} - \pi^2 \hat{f}(u_0,v_0) = 0 \tag{9.7}$$

which splits into

$$\hat{f}_1(u_0,v_0) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \frac{\hat{f}_1(u,v)}{(u-u_0)(v-v_0)} \tag{9.8}$$

$$\hat{f}_2(u_0,v_0) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \frac{\hat{f}_2(u,v)}{(u-u_0)(v-v_0)} \tag{9.9}$$

Relations (9.5-9.6) link real and imaginary parts of a function which is analytical in the half-plane $u' < 0$ and tends uniformly to zero as $|u| \to \infty$. However, their use is rather impractical if, let us say, the imaginary part should be recovered from the knowledge of the real part. Accordingly, it would be helpful to obtain *approximate relations* between real and imaginary parts at the same abscissa u_0. To this end, let us consider an integral of the type

$$I = \fint_{-\infty}^{+\infty} \frac{g(u)}{u-u_0}\, du = \int_{-\infty}^{0} \frac{g(u)}{u-u_0}\, du + \fint_{0}^{\infty} \frac{g(u)}{u-u_0}\, du \tag{9.10}$$

where the function $g(u)$ is explicitly assumed to be *even* and, without loss of generality, u_0 has been taken positive.

Now let

$$u = u_0 \exp(t) \quad , \quad u > 0 \tag{9.11}$$

$$u = -\, u_0 \exp(t), \quad u < 0 \tag{9.12}$$

Substituting (9.11-9.12) into (9.10), splitting the integration path of the second integral in $(-\infty, u_0^-)$, (u_0^+, ∞), integrating by part and rearranging terms results in

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \frac{d\, g(-u_0 \exp[t])}{dt}\, \ell n[1 + \exp(t)]dt \\ &\quad - \fint_{-\infty}^{+\infty} \frac{d\, g(u_0 \exp[t])}{dt}\, \ell n|1 - \exp(t)|dt \\ &= \fint_{-\infty}^{+\infty} \frac{dg}{dt}\, \ell n \coth \frac{|t|}{2}\, dt \end{aligned} \tag{9.13}$$

Result (9.13) is still exact under the only hypothesis that $g(u)$ is even. Note also that the sign of principal value can be omitted, since the integral of the

function $\ln \coth |t|/2$ from $-\infty$ to $+\infty$ is finite and equal to $\pi^2/2$. This function peaks rather sharply in the neighborhood of $t = 0$ and can be approximated by a Dirac pulse of area $\pi^2/2$. Then,

$$I \simeq \frac{\pi^2}{2} \left.\frac{dg}{dt}\right|_0 = \frac{\pi^2}{2} |u_0| \left.\frac{dg}{du}\right|_{u_0} \tag{9.14}$$

Assume now the function $\hat{f}(u)$ of (9.1) to be also free of zeros in the half-plane $u' \geq 0$ (the real axis is included). Then the logarithm of the absolute value of $\hat{f}(u)$ will be singularity-free in the half-plane $u' \geq 0$. Letting

$$\hat{f}(u) = |\hat{f}(u)| \exp[i\,\hat{\psi}(u)] \quad ; \quad \ln \hat{f}(u) = \ln |\hat{f}(u)| + i\,\hat{\psi}(u) \tag{9.15}$$

use of (9.5-9.6 and 9.14) shows that

$$\hat{\psi}(u) \simeq -\frac{\pi}{2} \frac{d \ln |\hat{f}(u)|}{d \ln |u/u_0|} \tag{9.15}$$

The above considerations can be applied to the far-field associated to an aperture of finite dimensions. As a matter of fact, eq. (5.6) shows that the radiation diagram of the aperture is given by

$$\beta \cos\theta\, \hat{\underline{E}}(u,v) = w\, \hat{\underline{E}}_t(u,v) - \hat{\underline{E}}_t \cdot \underline{k}_t\, \hat{z} \tag{9.16}$$

It is noted that $\hat{\underline{E}}_t$ is just the inverse Fourier transform of the tangent electric field in the aperture. Accordingly, $\hat{\underline{E}}_t$ is singularity-free in the half space $u' \geq 0$, $v' \geq 0$. The two factors $\underline{k}_t$ and w do not introduce any further singularity so that expression (9.16) still shares this property. However, the asymptotic behavior of (9.16) is different compared to that of $\hat{\underline{E}}_t$, so that, *in general*, real and imaginary parts of the radiation diagram are not a pair of Hilbert transforms. However, $\hat{\underline{E}}_t(u,v)$ decays more rapidly than $1/\sqrt{u^2+v^2}$ for $|u| \to \infty$ and/or $|v| \to \infty$ if a spatial tapering in the aperture is present. In this case, reasonings leading to Hilbert-transform properties of analytical functions apply, and *real and imaginary parts of the radiation diagram are a Hilbert-transform pair.*

APPENDIX

Exercises

1. Derive Hallen's integral equation for a thin wire antenna fed at its central point by an ideal current generator.

   Hint: Express the ϕ-component of magnetic field in terms of the vector potential. Substitute the expression of the vector potential in terms of the current distribution along the antenna. Then, specialize the result for $z = 0$ and take the source specification into account.

2. Consider an infinitely thin current filament extending along the z-axis from $-\infty$ to $+\infty$. The amplitude I of the current is constant along z and varies in time as $\exp(j\omega t)$. Derive the electromagnetic fields associated with this idealized antenna.

   Hint: Solve Maxwell's equations for the magnetic field and remember that $H_1^{(2)}(x) \simeq \frac{2j}{\pi x}$ as $x \to 0$.

3. Discuss the transient current response of a wire antenna of length 2a and diameter 2b driven by a gaussian voltage pulse $v(t) = V_0 \exp[-2t^2/T^2]$. The pulse "width" T is larger than $\pi\sqrt{\varepsilon_0\mu_0}\,a$. The antenna is surrounded by a homogeneous isotropic collisionless cold plasma whose relative permittivity is given by $\varepsilon(j\omega) = 1 - \frac{\omega_p{}^2}{\omega^2}$.

   Detailed computations are not required, although a clear statement of the problem is expected.

   Hint: Use the equivalent circuit representation for the input admittance of the antenna, truncated after the first dynamical term. For a better understanding of the branch-cut singularity, assume a small collision rate ν in the plasma, so that $\varepsilon(j\omega) = 1 - \frac{\omega_p{}^2}{\omega(\omega - j\nu)}$.

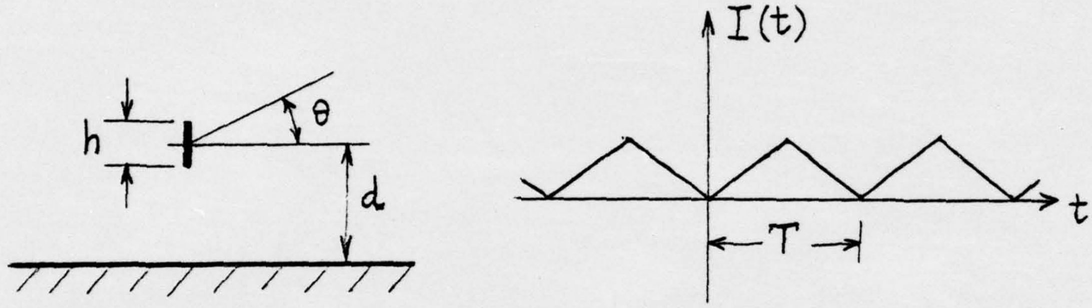
4. Consider an infinitely long cylindrical antenna of radius $\rho = a$. The antenna is assumed perfectly conducting and driven at $z = 0$ by a voltage $V\exp(j\omega t)$ applied to a δ-gap. Derive the modal expansion for the magnetic field generated by the antenna. Give the formal expression of the input admittance of the antenna.

   Hint: Assume the ϕ-component of magnetic field to have a z-dependence $\exp(-ju|z|)$, $0 \le u \le \infty$. Using Maxwell's equations, verify that the radial dependence is of type $H_1^{(2)}(\sqrt{k^2 - u^2}\,\rho)$. Use superposition to obtain the general expression for the magnetic field.

5. An elementary electric dipole of length h and capacitance C is driven by a current $I_0U(t)$, where $U(t)$ is the Heaviside step function. At large distance r there is a receiving elementary dipole of length h and capacitance C, parallel to the transmitting one. The receiving dipole is closed on a purely conducting load G. Compute the voltage across the load as a function of time.

<u>Hint</u>: Neglect radiation conductance of the dipoles. Take into account only the radiation term in the transmitted field.

6. The elementary electric dipole of figure is driven by triangular current pulses such that $cT \gg h$. The dipole is vertically located and at a distance d from a perfectly conducting plane. Find the direction of maximum and minimum (zero) radiation.

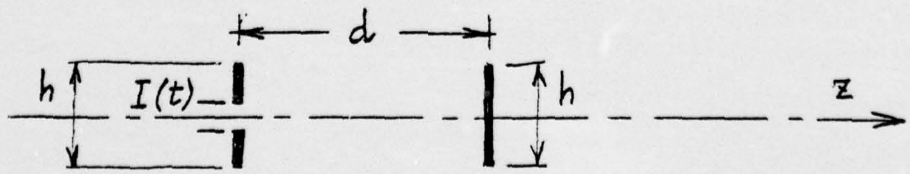


<u>Hint</u>: Apply the image theorem.

7. A wire antenna of length 2a and diameter 2b, $a \gg b$, is driven by a voltage pulse $v(t) = V_0\, f(t)$, $f(t) = 1$ for $0 \le t \le T$, $f(t) = 0$ otherwise. Assume cT large compared with b. Calculate the input current as a function of time.

<u>Hint</u>: Use the equivalent circuit truncated after the first dynamical term.

8. The elementary electric dipole of figure is driven by a current $I(t)$. A passive reflector is located at a distance d from the dipole. Assume the reflector to be in the near field of the transmitting dipole. Compute the field radiated along the end-fire direction $z > 0$.



<u>Hint</u>: Consider the reflector as a receiving antenna closed on a short-circuit.

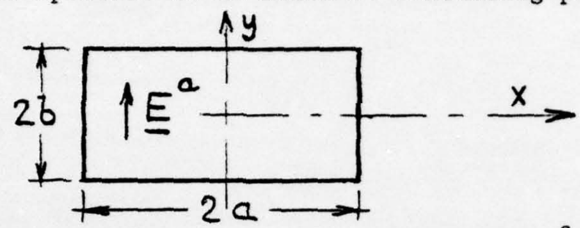
9. A loop antenna, of radius R and inductance L, is located parallel to a perfectly conducting plane and at a height d above it. The loop is driven by a guassian pulsed voltage

$$V(t) = V_0 \exp(-2t^2/T^2), \text{ such that } (cT)^2 \gg \pi R^2.$$

Calculate the transient far field.

Hint: Apply the image theorem.

10. Consider a rectangular aperture in an infinite conducting plane (see figure).



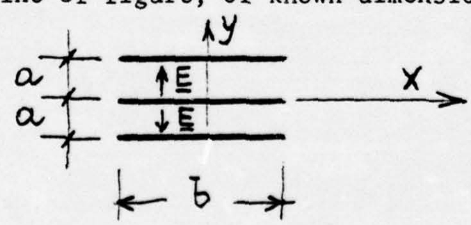
The aperture is illuminated by an electromagnetic field ($\underline{E}^a$,$\underline{H}^a$) of angular frequency ω, and the tangential electric field distribution across the aperture is given by

$$\underline{E}^a = E_0 \cos\left(\frac{\pi x}{2a}\right) \exp(ih\beta y)\, \hat{y}.$$

Determine the value of h for which the radiation diagram has a maximum at $\theta = 30°$ in the E-plane (θ is the angle with respect to the z-axis, and the E-plane is the $x = 0$ plane).

11. Consider the rectangular aperture of exercise #10. Determine the dimension 2b such that the main lobe aperture (angular distance between main lobe nulls) is equal to 10°. Determine the change in main lobe aperture due to a 10% change in the operating frequency.

12. Consider the strip-line of figure, of known dimensions and characteristic



impedance Z_0. The strip-line is excited with dominant TEM mode and a pulsed voltage $V = V_0\delta(z-ct)$ propagates along the line. Assume the electric field distribution to be uniform along x and y in the strip-line and compute the transient far field in the E-plane ($x = 0$ plane). Assume the thickness of conductors of the strip-line to be infinitesimally small.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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1. REPORT NUMBER: AFOSR - TR - 76 - 1128	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): TOPICS IN ADVANCED ANTENNA THEORY	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER: CL Report 76-2
7. AUTHOR(s): Giorgio Franceschetti	8. CONTRACT OR GRANT NUMBER(s): AF-AFOSR-72-2263-72
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Illinois at Chicago Circle, Communications Lab./Dept. Inform. Engrg., Box 4348, Chicago, Illinois 60680	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F, AF-9749-04, 974904
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20322	12. REPORT DATE: July 1976
	13. NUMBER OF PAGES: 87 (91p.)
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Antennas
Transient radiation
Dispersive environment
Transform properties

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

These lecture notes contain new research results in (i) antenna admittance and gap problem, (ii) transient radiation from antennas, and (iii) transform properties of radiated fields.

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